

# Introduction to Mean Field Games

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December 2022

# Motivations: Example 1 – Crowd Motion



Source: CGTN, Youtube

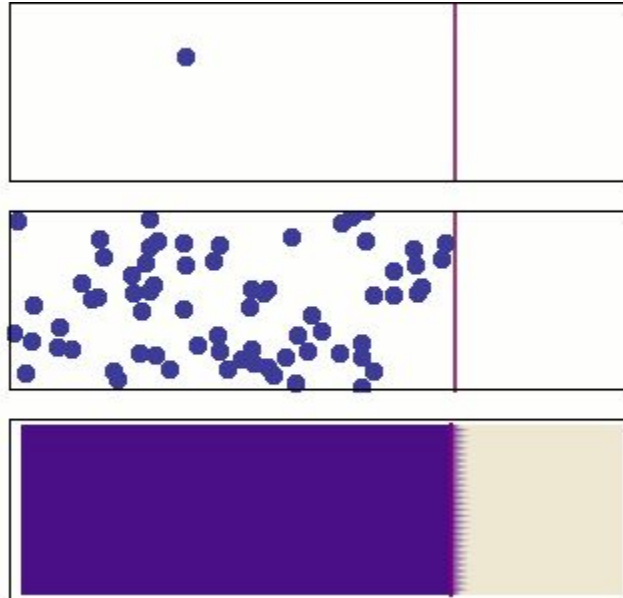
# Motivations: Example 1 – Crowd Motion



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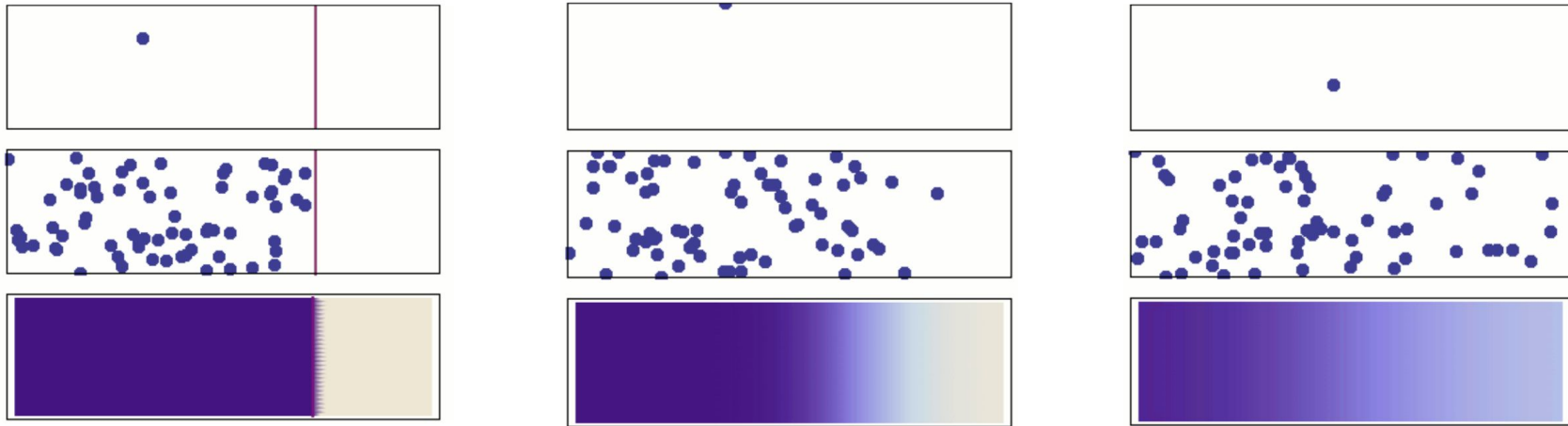
# Motivations: Example 1 – Crowd Motion

Macroscopic approximation



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Macroscopic approximation



# Motivations: Example 1 – Crowd Motion

## Incendie du 10 août 1903 dans le métro de Paris



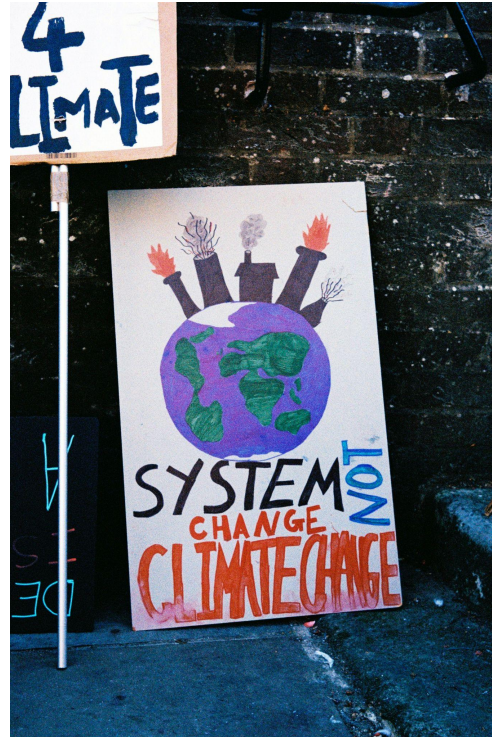


# Motivations: Example 2 – Economic Market



Source: Unsplash

# Motivations: Example 3 – Climate Change



Source: Unsplash



# Outline of the mini-course

1. MFG Models
2. Optimality Conditions
3. Numerical Methods

# 1. MFG models

# Outline

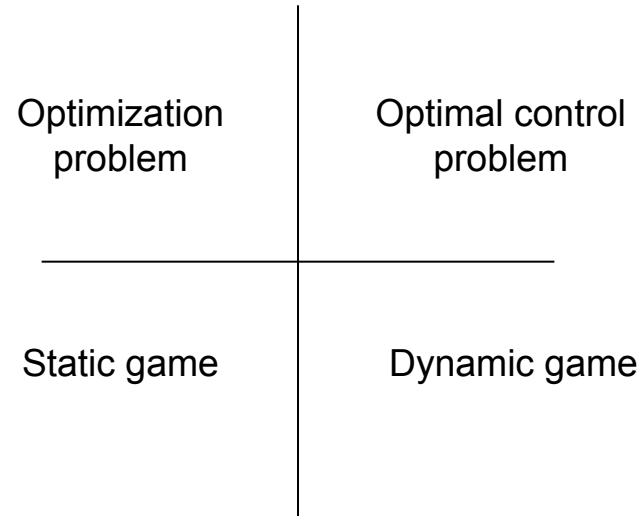
## 1. MFG Models

1.1 Static Setting

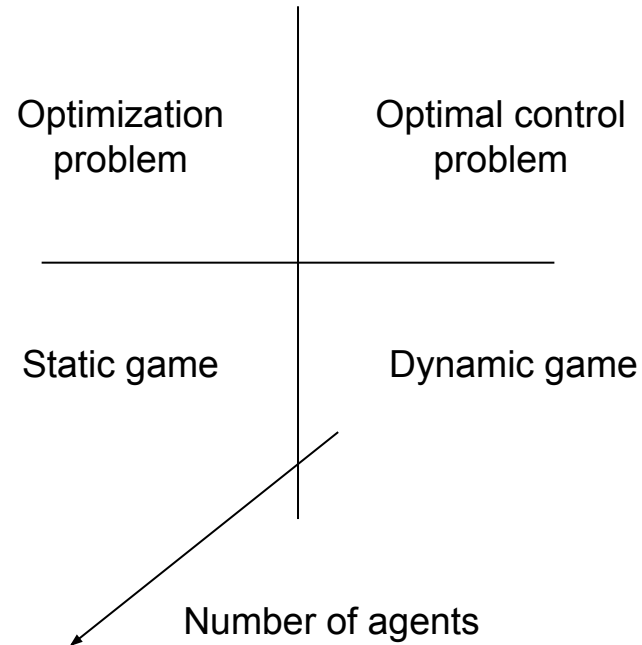
1.2 Social Optimum

1.3 Dynamic Setting

# Outline



# Outline





# 1.1 Static Setting

# 1.1.1 Static Setting: Finite-Population Game

# Notation

- $N$  players
- action space
- each player selects an action
- it induces a population profile of actions
- each player pays a cost
- goal of each player: minimize her own cost

# Nash Equilibrium

Main question: Is there a “stable configuration”?

Definition: Nash equilibrium (NE)

# An Example: Population Distribution

Population distribution game

Version 1: target position



# An Example: Population Distribution

Population distribution game

Version 2: attraction to the group

# An Example: Population Distribution

Population distribution game

Version 3: repulsion from the group

Example without Nash equilibrium?

# An Example: Population Distribution

Population distribution game

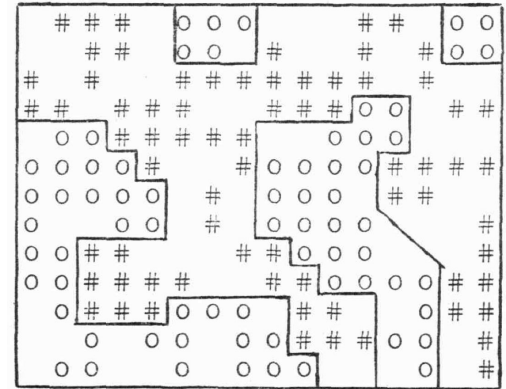
Version 4: spatial preferences + attraction to / repulsion from the group

# An Example: Population Distribution

## Schelling's model of segregation

- 2 types of agents
- each agent desires a fraction  $B$  of their neighbors to be of the same type
- repeat at each round:
  - check if the fraction is good
  - if not, relocate to a free location with a good fraction
- Schelling's result: threshold value  $B_{seg} \cong 1/3$  such that

If  $B > B_{seg}$ , then the iterations lead to *segregation*



# Mixed Strategies

Warning: a **pure NE does not always exist**

- $N$  players
- each player chooses a **mixed** strategy
- each player picks an action according to her strategy
- it induces population profiles of strategies and actions
- each player pays a cost
- goal for each player: minimize her own **average** cost

Note: the population distribution of actions is **random**



# Mixed Strategies

Warning: a **pure NE does not always exist**

Nash theorem: existence of **mixed NE**

**THEOREM 1.** *Every finite game has an equilibrium point.*

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior*. This book also contains a theory of  $n$ -person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an *equilibrium point* is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing “good strategies.”

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point.

# Mixed Strategies

Warning: a **pure NE does not always exist**

Nash theorem: existence of **mixed** NE

Proof based on fixed point theorem:

- Kakutani's fixed point theorem: based on the best-response mapping, which is in general a multi-valued mapping

**THEOREM 1.** *If  $x \rightarrow \Phi(x)$  is an upper semi-continuous point-to-set mapping of an  $r$ -dimensional closed simplex  $S$  into  $\mathbb{R}(S)$ , then there exists an  $x_0 \in S$  such that  $x_0 \in \Phi(x_0)$ .*

- Brouwer's fixed point theorem: *“Every continuous function from a convex compact subset  $K$  of a Euclidean space to  $K$  itself has a fixed point.”*

# Population Distribution Example: Mixed Strategies

Population distribution game

Version 1: target position

# Population Distribution Example: Mixed Strategies

Population distribution game

Version 2: attraction to the group

# Population Distribution Example: Mixed Strategies

Population distribution game

Version 3: repulsion from the group

What happens to the example without Nash equilibrium?



# Population Distribution Example: Mixed Strategies

Population distribution game

Version 4: spatial preferences + attraction to / repulsion from the group

# Large Population Games: $N \rightarrow +\infty$

- In many applications, the number of players is extremely large
- Intuitively,
  - each player has a **negligible impact** on the rest of the population
  - the population distribution of actions becomes **deterministic**
- This should simplify the analysis
- Can we formalize this intuition?

# Large Population Games: $N \rightarrow +\infty$

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  - each player has a **negligible impact** on the rest of the population
  - the population distribution of actions becomes **deterministic**
- This should simplify the analysis
- Can we formalize this intuition?
- Idea: **let  $N$  go to infinity** and study the problem we obtain in the limit
- Key assumptions: **homogeneity** and **anonymity**
- “**Mean field game**” paradigm [Lasry, Lions; Caines, Huang, Malhamé ~2006]

## 1.1.2 Static Setting: Mean Field Game

# Key Assumptions

To pass to the mean field limit, we assume **homogeneity** and **anonymity**

# Mean Field Game: Notation

- “Infinitely many” players
- each player chooses a (mixed) strategy
- each player picks an action according to the strategy
- it induces a population distribution of actions
- each player pays a cost
- goal for each player: minimize her own average cost

# Mean Field Game: Notation

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- each player chooses a (mixed) strategy
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- goal for each player: minimize her own average cost

## Key points:

- it is enough to understand the behavior of **one representative** player
- each player has **no influence** on the rest of the population

# Mean Field Nash Equilibrium

Definition: mean field Nash equilibrium (MFNE)

Fixed point formulation



# Population Distribution Example: MFG Viewpoint

Population distribution game

Version 1: target position

# Population Distribution Example: MFG Viewpoint

Population distribution game

Version 2: attraction to the group

# Population Distribution Example: MFG Viewpoint

Population distribution game

Version 3: repulsion from the group

# Population Distribution Example: MFG Viewpoint

Population distribution game

Version 4: spatial preferences + attraction to / repulsion from the group

⑧ “mean-field interactions” is more general than “interactions through the mean”

# Approximate Nash Equilibrium

Definition: Approximate Nash equilibrium in N-player game

Intuition: *An MFG equilibrium strategy provides an **approximate Nash equilibrium** in the corresponding finite-player game*

# Approximate Nash Equilibrium: Example

Example: interaction through the mean

Model:

**Assumption:** Mean field Nash equilibrium property

**Goal:**  $\varepsilon$ -Nash equilibrium for N-player game

# Approximate Nash Equilibrium: Example

Example: interaction through the mean

Proof sketch:

# Remarks

- Deterministic vs randomized decisions
- Discrete vs continuous action spaces
- Non-atomic anonymous games (continuum of players)



# Summary

Main takeaways so far

## 1.2 Social optimum

# Social Optimum: Static Setting

- Goal: minimize the social cost = average cost for the agents in the population
- N-agent social cost:
  
  
  
  
  
  
  
  
  
  
- Mean field social cost:

# Social Optimum vs Nash Equilibrium

- In general the two notions are different
- The socially optimal strategy is different from the Nash equilibrium policy
- Price of Anarchy

## A B S T R A C T

In a system where noncooperative agents share a common resource, we propose the price of anarchy, which is the ratio between the worst possible Nash equilibrium and the social optimum, as a measure of the effectiveness of the system. Deriving upper and lower bounds for this ratio in a model where several agents share a very simple network leads to some interesting mathematics, results, and open problems.<sup>2</sup>

# Social Optimum = Nash equilibrium: Example

In *some cases*, the two notions coincide.

Example: **Potential** MFG with cost:  $f(a, \nu) = \nabla F(\nu)(a)$

The average cost is:  $J(\pi, \nu) = \mathbb{E}_{a \sim \pi}[f(a, \nu)] = \sum_a \pi(a) \nabla F(\nu)(a) = \pi \cdot \nabla F(\nu)$

Assuming the potential **convex**, we have the equivalence:

$$\begin{aligned}
 \hat{\pi} \text{ is a NE} &\Leftrightarrow J(\pi, \hat{\pi}) - J(\hat{\pi}, \hat{\pi}) \geq 0, \quad \forall \pi \\
 &\Leftrightarrow (\pi - \hat{\pi}) \cdot \nabla F(\nu) \geq 0, \quad \forall \pi \\
 &\Leftrightarrow F(\pi) - F(\hat{\pi}) \geq 0, \quad \forall \pi \\
 &\Leftrightarrow \hat{\pi} \text{ is a minimizer of } F
 \end{aligned}$$

Example: entropy:  $F(\nu) = \sum_a \nu(a) \log(\nu(a))$

# Exercises

Ex. 1: Find a static MFG with exactly 2 pure NE. How many mixed NE are there?

Ex. 2: Find a static MFG with exactly 2 mixed social optima.

Ex. 3: Find a static MFG with a unique mixed NE and a unique mixed SO, such that their values are different. Same question with “such that their values are the same”.

# 1.3 Dynamic Setting

# 1.2.1 Dynamic Setting: Finite-Population Game



# Dynamic N-player Game

Main difference with static case: each player has a state which evolves in time

“Static” game

“Dynamic” game

# Dynamic N-player Game: Notation

- Time
- State space
- Action space
- One-step strategy (deterministic or mixed)
- Control or policy
- Player's state
- Population's state

# Dynamic N-player Game: Notation

We assume **homogeneity** and **anonymity**

- Player's dynamics
  
  
  
  
  
  
  
  
  
  
- Population's dynamics

# Dynamic N-player Game: Notation

- Running cost
- Terminal cost
- Total cost

# Nash Equilibrium

Definition: Nash equilibrium in dynamic N-player game

## 1.2.2 Dynamic Setting: Mean Field Game

# Dynamic Mean Field Game: Notation

- Time
- State space
- Action space
- One-step strategy (deterministic or mixed)
- Control or policy
- Player's state
- Population's state

# Dynamic Mean Field Game: Notation

- Player's dynamics:
- Population distribution dynamics:
- Mean field (MF) induced by a policy:



# Dynamic Mean Field Game: Notation

- Running cost
- Terminal cost
- Total cost
  
- Best response (BR) to a mean field:

# Example: Crowd Motion

- Dynamics:
- Cost:
  - Running cost:
    - cost to move (congestion):
    - discomfort (aversion):
  - Terminal cost:
    - spatial preference:

# Mean Field Nash Equilibrium

Definition: Mean field Nash equilibrium (MFNE) in a dynamic MFG

Fixed point formulation:

# Exercises

Ex. 1: Find a dynamic MFG such that:

(1) there is a unique NE

(2) given the equilibrium mean field sequence, there are multiple BR

## 1.2.3 Dynamic Setting: Continuous time & space

# MFG in Continuous Time and Space

For a (the) large(st) part, the MFG literature starts like this:

## INTRODUCTION

This paper is devoted to the analysis of second order mean field games systems with a local coupling. The general form of these systems is:

$$\begin{cases} (i) & -\partial_t \phi - A_{ij} \partial_{ij} \phi + H(x, D\phi) = f(x, m(x, t)) \\ (ii) & \partial_t m - \partial_{ij} (A_{ij} m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \\ (iii) & m(0) = m_0, \phi(x, T) = \phi_T(x) \end{cases} \quad (1)$$

Source: Cardaliaguet, P., Graber, P.J., Porretta, A. and Tonon, D., 2015. Second order mean field games with degenerate diffusion and local coupling. *Nonlinear Differential Equations and Applications NoDEA*, 22(5), pp.1287-1317.

In a nutshell, the probabilistic approach to the solution of the mean-field game problem results in the solution of a FBSDE of the McKean–Vlasov type

$$(3.1) \quad \begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)) dt + \sigma dW_t, \\ dY_t = -\partial_x H(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)) dt + Z_t dW_t, \end{cases}$$

with the initial condition  $X_0 = x_0 \in \mathbb{R}^d$ , and terminal condition  $Y_T = \partial_x g(X_T, \mathbb{P}_{X_T})$ .

Source: Carmona, R. and Delarue, F., 2013. Probabilistic analysis of mean-field games. *SIAM Journal on Control and Optimization*, 51(4), pp.2705-2734.

# Continuous Setting

*Why do we care* about continuous time & space?

# Continuous Setting

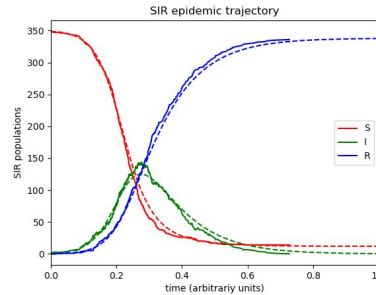
*Why do we care* about continuous time & space?

- Calculus!
- More natural for many applications
- Discretizing a continuous time/space process is not trivial



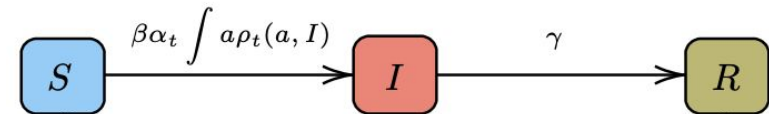
# Example 4 - Epidemics

## SIR model



ODE system: 
$$\begin{cases} \frac{dS}{dt} = -\frac{\beta IS}{N}, \\ \frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I, \\ \frac{dR}{dt} = \gamma I, \end{cases}$$

Basic reproduction number: 
$$R_0 = \frac{\beta}{\gamma}$$



where the **control**  $\alpha_t$  is a “contact” factor that depends on the individual’s behavior (e.g., socialization, wearing mask, ...)

The **running cost** encodes the individual’s preferences

It is also possible to include other aspects: vaccination, incentives, age structure, spatial movement, ...

Source: Wikipedia

Source: Kermack WO, McKendrick AG (1927). "A Contribution to the Mathematical Theory of Epidemics". Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character. 115 (772): 700–721.

Source: Aurell, A., Carmona, R., Dayanikli, G. and Lauriere, M., 2022. Optimal incentives to mitigate epidemics: a Stackelberg mean field game approach. SIAM Journal on Control and Optimization, 60(2), pp.S294-S322.

See also: Turinici, Hubert, et al.



# Example 5 - Flocking

## Cucker-Smale model

Position and velocity:

$$\begin{cases} x_i(t+1) = x_i(t) + v_i(t)\Delta t \\ v_i(t+1) = v_i(t) + \sum_j a_{i,j}(v_j(t) - v_i(t)) \end{cases}$$

with a matrix of interactions based on the positions:

$$a_{ij} = \eta(\|x_i - x_j\|^2) \quad \eta(y) = \frac{K}{(\sigma^2 + y)^\beta}$$

Source: Unsplash

Source: Cucker, F. and Smale, S., 2007. Emergent behavior in flocks. IEEE Transactions on automatic control, 52(5), pp.852-862.

Nourian, Caines & Malhamé'10:

Our aim in this work is to synthesize the collective behaviour of the set of agents from fundamental principles rather than to analyze this behaviour resulting from ad-hoc feedback laws. Hence the model in this paper may be regarded as a controlled game theoretic formulation of the uncontrolled C-S flocking model in which each agent, instead of responding to an ad-hoc algorithm, obtains its control law from a game theoretic Nash equilibrium depending upon its individual cost function and those of all other agents.

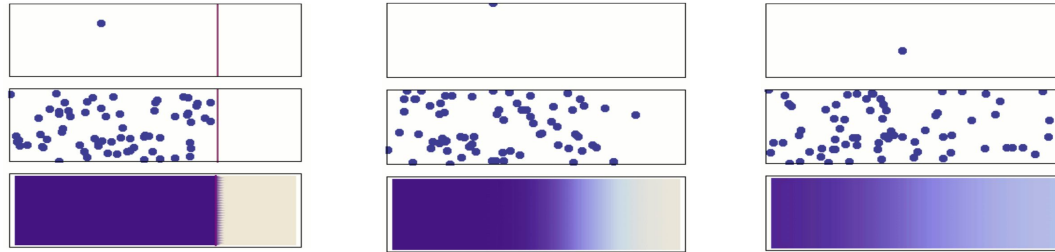
Velocity change = acceleration = **control**

**Running cost** penalizes deviation from neighbors' velocity:

$$\phi_i^{(N)}((x_i, v_i); (x, v)_{-i}) \triangleq \left\| \frac{1}{N} \sum_{j=1}^N w(\|x_i - x_j\|)(v_j - v_i) \right\|_Q^2$$

Source: Nourian, M., Caines, P.E. and Malhamé, R.P., 2010, September. Synthesis of Cucker-Smale type flocking via mean field stochastic control theory: Nash equilibria. In 2010 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton) (pp. 814-819). IEEE.

# Diffusion Model



- Particle's dynamics:  $dX_t = \sigma dW_t$
- Macroscopic distribution dynamics:  $\partial_t m(t, x) - \frac{\sigma^2}{2} \Delta m(t, x) = 0$
- Link with  $N$ -particle system: propagation of chaos [Kac'76]
  
- Note: We can also add a transport term (convection–diffusion equation)

# MFG in Continuous Time and Space

- Time
- Player's control (deterministic)
- Player's dynamics:  $dX_t = b(X_t, v(t, X_t), m(t))dt + \sigma dW_t, \quad X_0 \sim \mu_0$
- Population dynamics: **Kolmogorov-Fokker-Planck equation**

$$\partial_t m(t, x) - \frac{\sigma^2}{2} \Delta m(t, x) + \operatorname{div} \left( m(t, x) b(x, v(t, x), m(t)) \right) = 0, \quad m(0, x) = m_0(x)$$

# MFG in Continuous Time and Space

- Cost: dependence on the mean field
  - **non-local** (typically “regularizing” operator)
  - **local** (if the distribution has a density)

# Summary of Different Settings

Time \ Space	Discrete	Continuous
Discrete		
Continuous		

# Remarks and Extensions

- Discrete vs continuous: time, action and state spaces
- Controls:
  - Open-loop vs closed-loop
  - Deterministic vs randomized (pure vs mixed)
  - Interaction through the distribution of controls (“extended” MFG)
- Noise/perturbations:
  - With or without idiosyncratic noise (“first order” MFG)
  - With or without common noise
- Homogeneity: extension with multiple groups, major-minor, Stackelberg, ...
- Anonymity: multiple groups, graphon, ...

# Characterization of MFNE

Question:

*How can we **characterize** and **compute** mean field Nash equilibria?*



# Characterization of MFNE

Question:

*How can we **characterize** and **compute** mean field Nash equilibria?*

Answer:

*... in the rest of the mini-course.*

## 2. Optimality Conditions

# Outline

## 2. Optimality conditions

### 2.1 Introduction

### 2.2 Deterministic viewpoint

### 2.3 Stochastic viewpoint

## 2.1 Introduction

# Reminder: **Dynamic** MFNE

In the **dynamic** setting:

- Definition of MFNE
- Characterization?
  - Fixed point formulation
  - **Population** behavior
  - **Best response** characterization?

# Optimality conditions

*What do these equations mean?*

Large(st) part of the MFG literature starts like this:

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with the initial condition  $X_0 = x_0 \in \mathbb{R}^d$ , and terminal condition  $Y_T = \partial_x g(X_T, \mathbb{P}_{X_T})$ .

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## 2.2 Deterministic viewpoint

## 2.2.1 Deterministic Viewpoint: Discrete Setting



# Discrete Setting: Value Function

Definition: **Value function** of a representative player given a **mean field** sequence

Value of a state = sum of future costs, when starting from this state

# Discrete Setting: Bellman Equation

**Bellman equation** for the value function (**Dynamic Programming Principle**):

- Terminal time:
- Backward induction:

Recovering the optimal control from the value function:

# Discrete Setting: Forward-Backward System

**Coupled system:**

- **Forward** equation for the mean field:

$$\mu_{t+1}(x) = \sum_{x'} \mu_t(x') \sum_a \pi_t(a|x) p(x|x', a, \mu_t), \quad \mu_0 \text{ given}$$

- **Backward** equation for the value function:

$$V_t(x) = \min_a \mathbb{E}[f(X_t, A_t, \mu_t) + V_{t+1}(X_{t+1}) | X_t = x, A_t = a], \quad V_T(x) = g(x, \mu_T)$$

- Equilibrium policy:  $\pi$  satisfies: (1) is optimal against  $\mu$  and (2) generates  $\mu$

Challenge: *We cannot (fully) solve one equation before the other!*

# Discrete Setting: Existence and uniqueness?

**Existence:** generally based on **fixed point** formulation

Typically:

- **Banach/Picard** fixed point theorem
- **Brouwer/Schauder** fixed point theorem

# Discrete Setting: Existence and uniqueness?

**Uniqueness:** two cases:

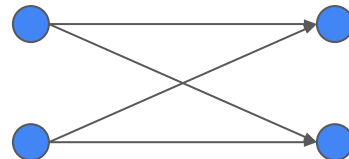
- **Contractivity:** uniqueness is a consequence of Banach fixed point theorem
- **Monotonicity:**  $V$  is monotone in  $L^2$  if: 
$$\int (V(x, m_1) - V(x, m_2))(m_1 - m_2)(x) dx \geq 0$$
  - Typical setting:  $b(x, a, \mu) = b(x, a)$ ,  $f(x, a, \mu) = \tilde{f}(x, a) + V(x, \mu)$
  - Example: crowd motion (control = velocity) with cost = movement + crowd aversion

# Discrete Setting: Example of Existence Proof

Sketch of existence proof:  $\Phi : \pi \xrightarrow{\text{MF}} \tilde{\mu} \xrightarrow{\text{BR}} \tilde{\pi}$

A simple model:

- $\mathcal{X} = \{0, 1\}$ ,  $\mathcal{A} = \{-1, 0, 1\}$ ,  $T = 1$
- $X_{t+1} = X_t + A_t$  with walls at  $x = -1, 2$
- $f(x, a, \mu) = 0$ ,  $g(x, \mu) = |x - \bar{\mu}|$ ,  $\bar{\mu} = \text{mean of } \mu$
- $\mu_0 = (\frac{1}{2}, \frac{1}{2})$



# Discrete Setting: Example of Existence Proof

**Step 1:** Convexity and compactness

**Step 2:** Continuity of  $\Phi$

**Step 2.a:** Continuity of MF?

# Discrete Setting: Example of Existence Proof

**Step 1:** Convexity and compactness

**Step 2:** Continuity of  $\Phi$

**Step 2.b:** Continuity of BR?



## 2.2.2 Deterministic Viewpoint: Continuous Setting

# Continuous Setting: Value Function

Definition: **Value function** of a representative player given a **mean field** flow

Value of a state = sum of future costs, when starting from this state

Dynamic Programming Principle?

# Continuous Setting: HJB Equation

Hamiltonian:  $H(x, m, p) = \max_a -L(x, a, m, p), \quad L(x, a, m, p) = f(x, a, m) + b(x, a, m) \cdot p$

Hamilton-Jacobi-Bellman equation:

$$-\partial_t u(t, x) - \frac{\sigma^2}{2} \Delta u(t, x) + H(x, m(t), \nabla u(t, x)) = 0, \quad u(T, x) = g(x, m(T))$$

Recovering the optimal control:

# Continuous Setting: Forward-Backward System

**Coupled system:**

- **Forward** equation for the mean field:

$$\partial_t m(t, x) - \frac{\sigma^2}{2} \Delta m(t, x) - \operatorname{div}(m(t, x) H_p(x, m(t), \nabla u(t, x))) = 0, \quad m(0, x) = m_0(x)$$

- **Backward** equation for the value function:

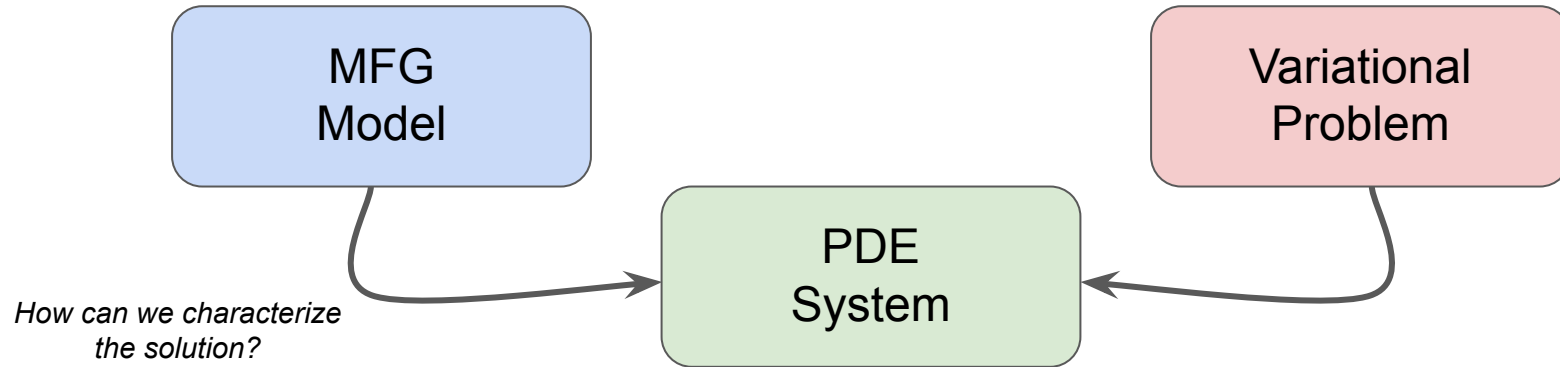
$$-\partial_t u(t, x) - \frac{\sigma^2}{2} \Delta u(t, x) + H(x, m(t), \nabla u(t, x)) = 0, \quad u(T, x) = g(x, m(T))$$

Challenge: *We cannot (fully) solve one equation before the other!*

# Existence and Uniqueness of MFNE

- Existence: generally obtained by applying a fixed point theorem, such as:
  - Banach fixed point theorem: typically applicable under “smallness” conditions (small time or small Lipschitz constants); gives uniqueness too
  - Schauder fixed point theorem: applicable more generally; does not yield uniqueness
  - Compactness can be challenging
- Uniqueness:
  - Contractivity (application of Banach fixed point theorem; “smallness” assumptions)
  - Monotonicity condition (Lasry & Lions; “structural” assumption)

# Variational MFGs



In some cases, the MFG PDE system can be interpreted as the optimality conditions for a variational problem (See e.g. Lasry & Lions'06, Cardaliaguet & Graber'15, ...)

# MFG with Common Noise

- Common noise: randomness affecting the whole population
- Example: extra Brownian motion common to all the players
- Then the two PDEs become **stochastic** PDEs

$$\left\{ \begin{array}{l} d_t u_t = \left\{ -(1 + \beta) \Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta} \operatorname{div}(v_t) \right\} dt + v_t \cdot \sqrt{2\beta} dW_t \\ \quad \text{in } [0, T] \times \mathbb{T}^d, \\ d_t m_t = \left[ (1 + \beta) \Delta m_t + \operatorname{div}(m_t D_p H(m_t, Du_t)) \right] dt - \operatorname{div}(m_t \sqrt{2\beta} dW_t), \\ \quad \text{in } [0, T] \times \mathbb{T}^d, \\ u_T(x) = G(x, m_T), \quad m_0 = m_{(0)}, \quad \text{in } \mathbb{T}^d \end{array} \right.$$

See: Cardaliaguet, P., Delarue, F., Lasry, J.M. and Lions, P.L., 2019. The master equation and the convergence problem in mean field games:(ams-201). Princeton University Press.

# MFG with Common Noise

- Common noise: randomness affecting the whole population
- Example: extra Brownian motion common to all the players
- Convergence analysis (as  $N \rightarrow \infty$ ) based on the **Master equation**

$$\left\{ \begin{array}{l} -\partial_t U - (1 + \beta)\Delta_x U + H(x, D_x U) \\ \quad - (1 + \beta) \int_{\mathbb{R}^d} \operatorname{div}_y [D_m U] dm(y) + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U) dm(y) \\ \quad - 2\beta \int_{\mathbb{R}^d} \operatorname{div}_x [D_m U] dm(y) - \beta \int_{\mathbb{R}^{2d}} \operatorname{Tr} [D_{mm}^2 U] dm \otimes dm = F(x, m) \\ \quad \text{in } [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \\ U(T, x, m) = G(x, m) \quad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \end{array} \right.$$

See: Cardaliaguet, P., Delarue, F., Lasry, J.M. and Lions, P.L., 2019. The master equation and the convergence problem in mean field games:(ams-201). Princeton University Press.



# Exercises

Ex. 1: For the following drift and running cost function, write the KFP equation, the Hamiltonian and the HJB equation:

$$\text{LQ :} \quad b(x, a, m) = Ax + Ba + \bar{A}\bar{m}^2, \quad f(x, a, m) = Qx^2 + Ra^2 + \bar{Q}\bar{m}^2, \quad \bar{m} = \int \xi m(\xi) d\xi$$

$$\text{Congestion} \quad b(x, a, m) = a, \quad f(x, a, m) = m(x)|a|^2$$

$$\text{Aversion :} \quad b(x, a, m) = a, \quad f(x, a, m) = |a|^2 + m(x)$$

Ex. 2: Derive optimality conditions for the social optimum problem.

# Exercises

Ex. 3 [Bogachev, Krylov, Röckner, Shaposhnikov; Thm 9.8.41]:

Consider the MFG PDE system:

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2} |\nabla u|^2 = F(x, \mu_t), & \mathbb{R}^d \times [0, T), \\ \partial_t \mu_t - \Delta \mu_t - \operatorname{div}(\mu_t \nabla u) = 0, & \mathbb{R}^d \times (0, T], \end{cases}$$

with  $u(x, T) = G(x, \mu_T)$ ,  $\mu_0 = \nu$ .

Part 1: Write the player's dynamics and the cost function.

Part 2: Show existence of a classical solution, assuming:

- $\nu$  is a probability distribution on  $\mathbb{R}^d$  with finite second moment
- $F, G: \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}$  are bounded and Lipschitz

## 2.3 Stochastic viewpoint

## 2.3.1 Stochastic Viewpoint: From Bellman's Principle

# Reminder: Value Function

- **Value function** of a representative player given a **mean field** sequence
- **Hamilton-Jacobi-Bellman** equation:

$$-\partial_t u(t, x) - \frac{\sigma^2}{2} \Delta u(t, x) + H(x, m(t), \nabla u(t, x)) = 0, \quad u(T, x) = g(x, m(T))$$

- Actually in practice, we do not really need to know  $u$  **everywhere**

## Reminder: Value Function

- We want to know  $u$  and the control along the path of  $X$
- Introduce  $Y_t = u(t, X_t)$
- Dynamics of  $Y$ :

$$dY_t = \frac{d}{dt}u(t, X_t)$$

$$= \left[ \partial_t u(t, X_t) + \frac{\sigma^2}{2} \Delta u(t, X_t) \right] dt + \partial_x u(t, X_t) dX_t$$

(by Itô's lemma)

$$= -f(X_t, A_t^*, \hat{\mu}_t) dt + Z_t dW_t$$

(by HJB equation)

## 2.3.2 Stochastic Viewpoint: From Pontryagin's Principle

## Another point of view: Co-state

- Assume  $X$  has a deterministic evolution:  $\dot{x}_t = b(x_t, a_t)$ ,  $x_0$  given
- Hamiltonian:  $H(x, p) = \max_a -L(x, a, p)$ ,  $L(x, a, p) = f(x, a) + b(x, a) \cdot p$
- **Pontryagin's maximum principle:**

$$\left\{ \begin{array}{l} \dot{x}_t^* = b(x_t^*, a_t^*), \\ \dot{y}_t^* = -\partial_x L(x_t^*, a_t^*, y_t^*), \\ a_t^* = \operatorname{argmax}_a -L(x_t^*, a, y_t^*) \end{array} \right. \quad \begin{array}{l} x_0^* \text{ given} \\ y_T^* = \partial_x g(x_T^*) \end{array}$$

- In fact,  $Y_t$  can be interpreted as  $\partial_x u(t, X_t)$



## Another point of view: Co-state

- If  $X$  satisfies an SDE:  $dX_t = b(X_t, A_t)dt + \sigma dW_t$ ,  $X_0 \sim m_0$
- Hamiltonian:  $H(x, p) = \max_a -L(x, a, p)$ ,  $L(x, a, p) = f(x, a) + b(x, a) \cdot p$
- **Stochastic maximum principle:**

$$\left\{ \begin{array}{l} dX_t^* = b(X_t^*, A_t^*)dt + \sigma dW_t, \\ dY_t^* = -\partial_x L(X_t^*, A_t^*, Y_t^*)dt + Z_t^* dW_t, \\ A_t^* = \operatorname{argmax}_a -L(X_t^*, a, Y_t^*) \end{array} \right. \quad \begin{array}{l} X_0 \sim m_0 \\ Y_T^* = \partial_x g(X_T^*) \end{array}$$

- In fact,  $Y_t$  can be interpreted as  $\partial_x u(t, X_t)$

## Another point of view: Co-state

- If  $X$  satisfies a **mean field** SDE:  $dX_t = b(X_t, A_t, \hat{\mu}_t)dt + \sigma dW_t$ ,  $X_0 \sim m_0$
- Hamiltonian:  $H(x, m, p) = \max_a -L(x, a, m, p)$ ,  $L(x, a, m, p) = f(x, a, m) + b(x, a, m) \cdot p$
- **Stochastic maximum principle:**

$$\begin{cases} dX_t^* = b(X_t^*, A_t^*, \hat{\mu}_t)dt + \sigma dW_t, & X_0 \sim m_0 \\ dY_t^* = -\partial_x L(X_t^*, A_t^*, \hat{\mu}_t, Y_t^*)dt + Z_t^* dW_t, & Y_T^* = \partial_x g(X_T^*, \hat{\mu}_T) \\ A_t^* = \operatorname{argmax}_a -L(X_t^*, a, \hat{\mu}_t, Y_t^*) \end{cases}$$

- Note: for the equilibrium, we need to include the consistency condition for the **MF**
- In fact,  $Y_t$  can be interpreted as  $\partial_x u(t, X_t)$

# FBSDE Systems

In both cases (from Bellman or Pontryagin's principles), we get an instance of a McKean-Vlasov forward-backward SDEs (MKV-FBSDE):

$$\left\{ \begin{array}{l} dX_t = B(X_t, Y_t, Z_t, \mu_t)dt + \sigma dW_t, \quad X_0 \sim m_0 \\ dY_t = F(X_t, Y_t, Z_t, \mu_t)dt + Z_t dW_t, \quad Y_T = G(X_T, \mu_T) \\ \mu_t = \mathcal{L}(X_t) \end{array} \right.$$

- Analysis: existence, uniqueness, ...
- Link with Master equation
- See Carmona & Delarue's book

# Exercises

Ex. 1: For the following drift and running cost function, write the MKV FBSDE system:

$$\text{LQ :} \quad b(x, a, m) = Ax + Ba + \bar{A}\bar{m}^2, \quad f(x, a, m) = Qx^2 + Ra^2 + \bar{Q}\bar{m}^2, \quad \bar{m} = \int \xi m(\xi) d\xi$$

$$\text{Congestion} \quad b(x, a, m) = a, \quad f(x, a, m) = m(x)|a|^2$$

$$\text{Aversion :} \quad b(x, a, m) = a, \quad f(x, a, m) = |a|^2 + m(x)$$

Ex. 2: Derive an FBSDE system for the social optimum problem.

# 3. Numerical Methods

# Outline

## 3. Numerical Methods

### 3.1 “Learning” in MFGs

### 3.2 Solving Forward-Backward Equations

## 3.1 “Learning” in MFGs

# Learning in Games

- Several different meanings
- Algorithmic game theory, economics, ...
- Recent successes of (machine) learning for games (Chess, Go, Stratego, ...)
- Connection with Multi-Agent Reinforcement Learning (MARL)
- Most existing methods are not directly suitable for large populations
- MFG can be a way to **scale-up** in terms of **number of agents**



# Learning in MFG

- Fixed point formulation
- Fixed point iterations
  
- Converges if strict contraction, but otherwise might fail (oscillations, ...)

# Learning in MFG

- To **stabilize** the learning, several variants have been introduced
  - **Averaging** over past iterations
    - ↪ Fictitious Play, Online Mirror Descent, ...
  - **Smoothing** the policy
    - ↪ Softmax/Boltzmann policies, ...
  - **Regularization** through the cost
    - ↪ Entropic penalty, ...
  - ...

# Example: Fictitious Play

For simplicity, we consider a static MFG

- 2 cities connected by 2 roads; players = drivers; action = choice of road
- Initialization: everyone chooses road 1  $\Rightarrow$  traffic jam
- Iteration 1: everyone chooses road 2  $\Rightarrow$  traffic jam
- ...
- No convergence

# Example: Fictitious Play

For simplicity, we consider a static MFG

- If players react to the average of past distributions  
⇒ convergence of the mean field (*but maybe not the last iterate*)

$$\begin{cases} a^{(k)} = \text{BR}(\nu^{(k)}) \\ \nu^{(k+1)} = \frac{k}{k+1}\nu^{(k)} + \frac{1}{k+1}\delta_{a^{(k)}} \end{cases}$$

# Best Response Update

In a dynamic setting, at each iteration, we need to:

- Update the mean field
- Update the policy/control
  - optimality conditions
    - deterministic
    - stochastic
  - model-free learning (RL)

## 3.2 Solving Forward-Backward Equations

# Numerical methods

Two main questions:

- Approximation, e.g.
  - discretization (time, space, population)
  - parameterization (grid, neural networks, ...)
  - ...
- Computation, e.g.
  - fixed point algorithms
  - optimization methods
  - ...

# Deterministic Approach

- Finite time and space
- Continuous time and finite space
  - numerical schemes for the time variable
- Continuous time and finite space:
  - traditional numerical schemes for PDEs (finite differences, finite elements, ...)
  - neural network-based methods (e.g., DGM method by Sirignano & Spiliopoulos)
- Alternatives to fixed-point:
  - e.g. solving the whole system at once [Achdou, Capuzzo-Dolcetta]



# Example: Finite Difference Scheme

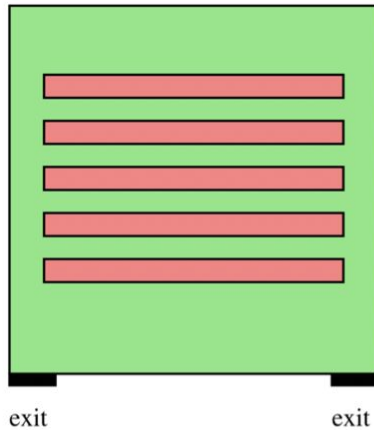
Finite difference scheme for the PDE system [Achdou & Capuzzo-Dolcetta]

$$\left\{ \begin{array}{l} (D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}) = 0, \\ M_0^n = M_{N_h}^n, \\ M_i^0 = \bar{m}_0(x_i), \end{array} \right. \quad \begin{array}{l} i \in \{0, \dots, N_h\}, n \in \{0, \dots, N_T - 1\}, \\ n \in \{1, \dots, N_T\}, \\ i \in \{0, \dots, N_h\}, \end{array}$$

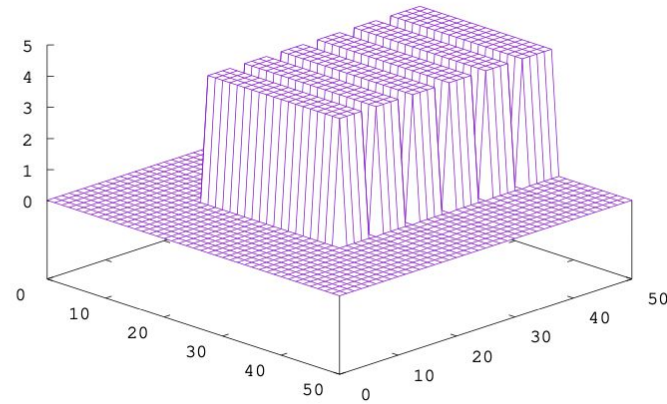
$$\left\{ \begin{array}{l} -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}(x_i, [\nabla_h U^n]_i) = f_0(x_i, M_i^{n+1}), \\ U_0^n = U_{N_h}^n, \\ U_i^{N_T} = \phi(M_i^{N_T}), \end{array} \right. \quad \begin{array}{l} i \in \{0, \dots, N_h\}, n \in \{0, \dots, N_T - 1\}, \\ n \in \{0, \dots, N_T - 1\}, \\ i \in \{0, \dots, N_h\}. \end{array}$$

# Example: Finite Difference Scheme

Crowd motion with congestion



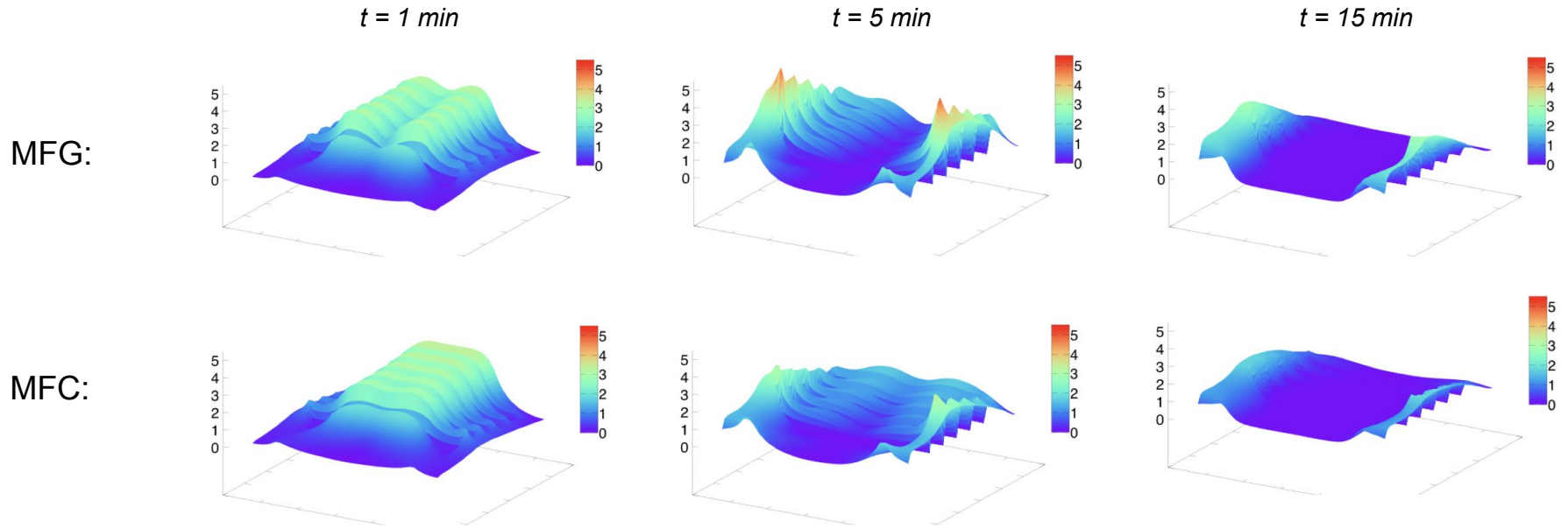
*Room*



*Initial density*

# Example: Finite Difference Scheme

Crowd motion with congestion: Nash equilibrium vs Social optimum



Source: Achdou, Y. and Laurière, M., 2020. Mean field games and applications: Numerical aspects. Mean field games, pp.249-307.

Original paper: Achdou, Y. and Capuzzo-Dolcetta, I., 2010. Mean field games: numerical methods. SIAM Journal on Numerical Analysis, 48(3), pp.1136-1162.

# Stochastic Viewpoint

- Approximation of the forward SDE (state process):
  - Euler-Maruyama scheme [Bossy, Talay; ...]
  - Need to take into account the mean field interactions
- Approximation of the **backward SDE** (value function or its derivative)
  - Tree based methods [Chassagneux, Crisan, Delarue; ...]
  - Regression Monte Carlo methods [Pham et al.; ...]

# Stochastic Viewpoint

- Several methods using neural networks, motivated by high dimensionality
  - Direct approach to solve control problems
  - DeepBSDE [E, Han, Jentzen]
  - DBDP [Pham, Warin et al.]
  - For standard BSDE first, and have been adapted to the MF setting

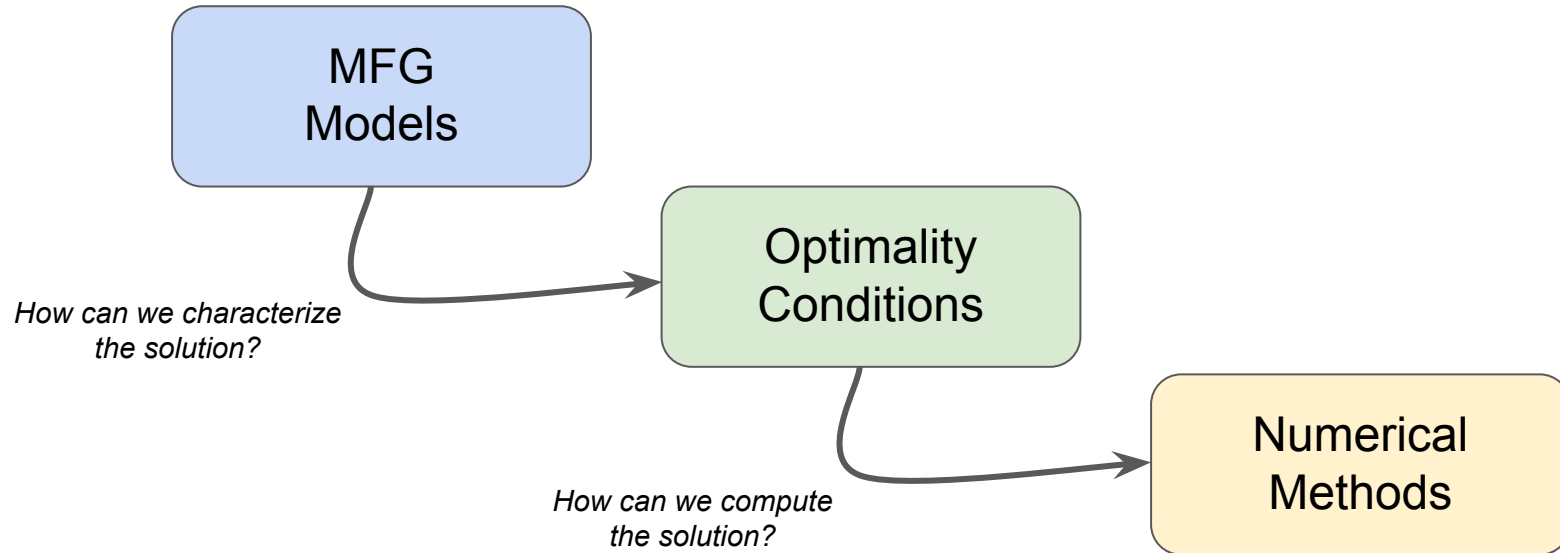
# Summary and Perspectives

# Summary

## What MFGs are NOT:

- agent-based model / multi-agent system  
→ scalability issues when  $N$  is large
- (mean-field) dynamical system  
→ no game or optimization
- non-atomic anonymous game  
→ continuum of players

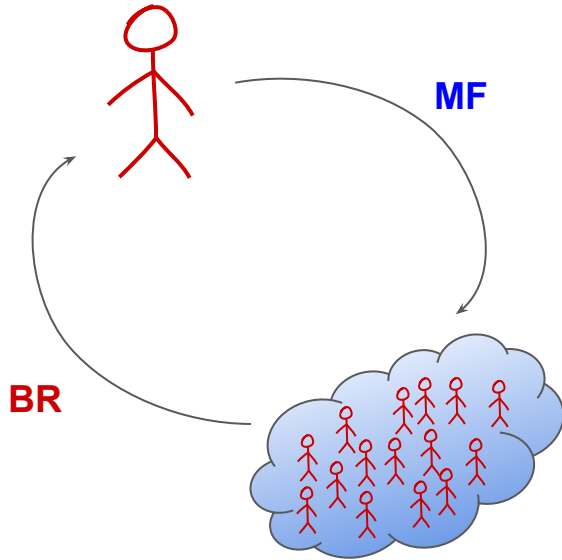
# Summary





# Summary

## 1. Mean-field Nash equilibrium (mean-field social optimum)



- Mean-field approximation:
  - Simplifies the analysis
  - Provides an approximate N-player equilibrium
- MFG as a fixed point problem
- MFC as an optimization problem

# Summary

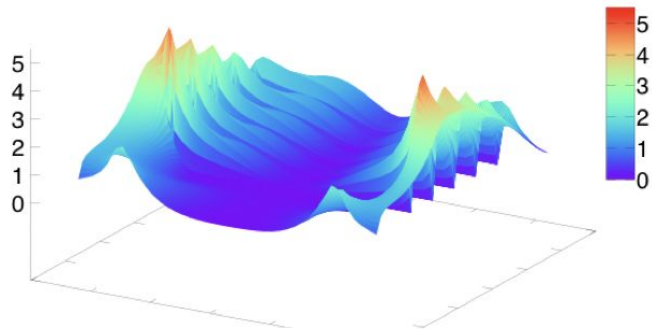
## 2. Optimality conditions

{ Forward equation for the mean-field state or the player's state  
Backward equation for the value function or the backward process

- Necessary (and sometimes sufficient) conditions of “optimality” (in fact “equilibrium”)
- PDE / SDE viewpoint
- Generally fully coupled system
- Existence proof e.g. through fixed point argument

# Summary

## 3. Numerical methods



- Approximation
- Algorithms
- Methods inspired by “learning” algorithms
- Key tool for real-world applications of MFGs

# Perspectives: Some Research Directions

- MFG Models for new applications
  - complex models (dynamics and cost function)
  - beyond perfect homogeneity and anonymity
- Rigorous justification of the mean field limit
  - from MFG equilibria to N-player (approximate) equilibria
  - convergence of N-player Nash equilibria to MFG (rate, large deviations, ...)
- Derivation and analysis of the optimality conditions
  - existence/uniqueness
  - regularity
- Numerical methods
  - developing & analyzing machine learning methods
- Data-driven approaches
  - combining MFG paradigm with data for real-world predictions

# Some Extra References

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