Learning Methods in Mean Field Games
Parts 1 & 2

Mathieu LAURIÈRE

NYU Shanghai

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Questions, comments or suggestions are most welcome.

Based on joint works with many people, including:

Andrea Angiuli, Olivier Bachem, Tamer Basar, Theophile Cabannes, René Carmona, Gökçe Dayanikli, Romuald Élie, Jean-Pierre Fouque, Matthieu Geist, Maximilien Germain, Sertan Girgin, Kenza Hamidouche, Ruimeng Hu, Ayush Jain, Alec Koppel, Raphael Marinier, Paul Muller, Rémi Munos, Julien Pérolat, Sarah Perrin, Huyên Pham, Olivier Pietquin, Georgios Piliouras, Mark Rowland, Zongjun Tan, Karl Tuyls, Muhammad Aneeq uz Zaman, . . .

as well as other people's works

Outline

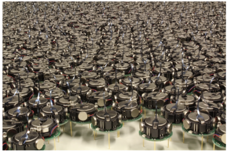
Motivations

Flocking

Crowd motion

Traffic flow

Collective AI

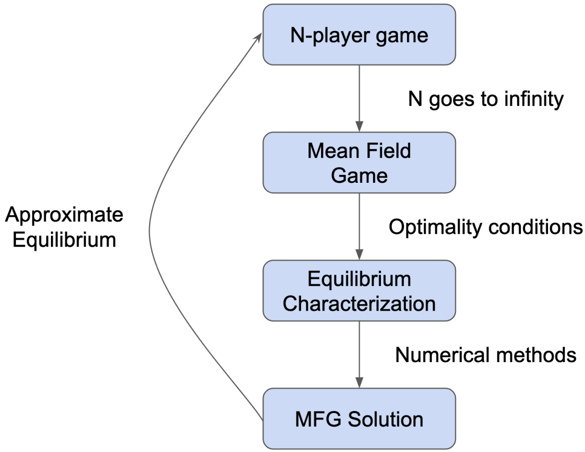


[Image credits: Unsplash, Wikimedia Commons (Kilobots)]

Some other existing approaches *("What MFGs are* ***not****")*

- Dynamical systems:
  - describe the dynamics of one or many agents, sometimes mean field
  - but usually **no rationality** (optimization)
- Agent based models (ABM):
  - "Agent-based models are a kind of **microscale model** that simulate the simultaneous operations and interactions of multiple agents in an attempt to re-create and predict the appearance of complex phenomena. "
  - "Individual agents are typically characterized as **boundedly rational**, presumed to be acting in what they perceive as their own interests, such as reproduction, economic benefit, or social status, using heuristics or simple decision-making rules." (Wikipedia)
- Game theory
  - optimization aspects
  - notion of Nash equilibrium, social optimum, ...
  - but usually limited to a **finite (small) number of agents**
- Evolutionary game theory (EGT)
  - "application of game theory to evolving populations in biology"
  - "an evolutionary version of game theory **does not require players to act rationally** – only that they have a strategy" (Wikipedia)
- Non-atomic anonymous games
  - continuum of rational players; each player has her **own index** and own strategy
  - mostly limited to static games; difficulties for dynamic, stochastic games

MFG paradigm in a nutshell



Some References

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  - Pierre-Louis Lions' lectures at Collège de France (`https://www.college-de-france.fr/`)
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- Laurière, M., Perrin, S., Geist, M., & Pietquin, O. (2022). Learning mean field games: A survey. arXiv preprint arXiv:2205.12944.

MFG & (R)L: Motivations

Main motivation: real-world applications require methods for large-scale problems

- Scaling up **population size** $\rightarrow$ **Mean Field Games**
  - Initial papers: Lasry & Lions; Caines, Huang & Malhamé (2006-2007)
  - Books: Bensoussan, Frehse & Yam; Carmona & Delarue; ...
- Scaling up **environment complexity** $\rightarrow$ (model-free) **Reinforcement Learning**
  - Book: Sutton & Barto; ...
  - Applications: Robotics, language processing, games, ...

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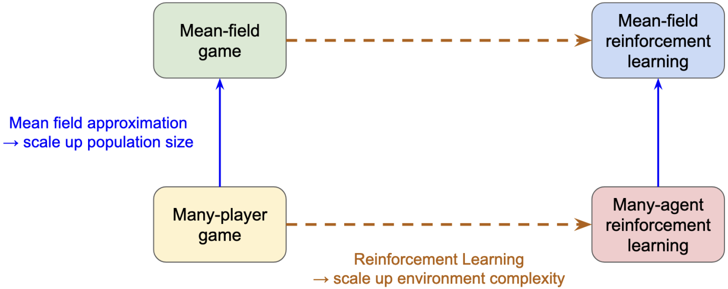

Motivations behind this overview

Rapidly growing literature

Goal: overview of the landscape & codes to make this topic more easily accessible

A few key aspects:

1. Problem setting

   $\rightarrow$ *continuous / discrete time & space, ...*

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   $\rightarrow$ *learning solution with "ideal" updates*

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   $\rightarrow$ *learning solution with model-free updates*
5. Implementation

   $\rightarrow$ *code samples (OpenSpiel, ...)*

Learning

Recent successes of learning in games, e.g.:

Go [SHM+16, SSS+17, SHS+18], Chess [CHJH02], Checkers [SBB+07], Hex [ATB17], Starcraft II [VBC+19], poker games [BS17, BS19, MSB+17, BBJT15], Stratego [MLFB20], . . .

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At least **two interpretations** of "learning":

- Game theory, economics, ...:
  Fudenberg & Levine [FL09][1]: *"The theory of learning in games [...] examines how, which, and what kind of equilibrium might arise as a consequence of a long-run nonequilibrium process of learning, adaptation, and/or imitation"*

- Machine Learning, Reinforcement Learning, ...:
  Mitchell [M$^+$97][2]: *"A computer program is said to learn from experience E with respect to some class of tasks T and performance measure P if its performance at tasks in T, as measured by P, improves with experience E."*

[1] Fudenberg, D., & Levine, D. K. (2009). Learning and equilibrium. *Annu. Rev. Econ.*, 1(1), 385-420.

[2] Mitchell, T. M. (1997). *Machine Learning*. New York: McGraw-Hill. ISBN: 978-0-07-042807-2

Outline

N-Player Stochastic Differential Game

For now, continuous time and continuous space:

- N players
- Player i's state is $X_t^i \in \mathbb{R}^d$
- with dynamics:

$$dX_t^i = b(t, X_t^i, \alpha_t^i, \mu_t^N)dt + \sigma dW_t^i, \qquad X_0^i \sim m^0$$

- W^i is an idiosyncratic (individual) noise, independent from other W^j's
- The empirical state distribution is: $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$

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- The empirical state distribution is: $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$
- Instantaneous cost function f and terminal cost function g
- Goal for player i: minimize over α^i the total expected cost:

$$J(\alpha^i, \alpha^{-i}) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \alpha_t^i, \mu_t^N)dt + g(X_T^i, \mu_T^N)\right]$$

N-Player Stochastic Differential Game: Solution Concepts

Two concepts:

- **Nash equilibrium** $(\hat{\alpha}^1, \dots, \hat{\alpha}^N)$: for all $i = 1, \dots, N$ and all α^i,

$$J(\hat{\alpha}^i, \hat{\alpha}^{-i}) \leq J(\alpha^i, \hat{\alpha}^{-i})$$

  $\rightarrow$ no incentive for unilateral deviations

  $\rightarrow$ **fixed point** problem

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- **Social optimum** $(\alpha^{*1}, \dots, \alpha^{*N})$: for all $i = 1, \dots, N$ and all $(\alpha^1, \dots, \alpha^N)$,

$$\bar{J}(\alpha^{*1}, \dots, \alpha^{*N}) = \frac{1}{N} \sum_{i=1} J(\alpha^{*i}, \alpha^{*-i}) \leq \bar{J}(\alpha^1, \dots, \alpha^N) = \frac{1}{N} \sum_{i=1} J(\alpha^i, \alpha^{-i})$$

  $\rightarrow$ no incentive for joint deviations
  $\rightarrow$ **optimization** problem

In general, they are different, which leads to the notion of Price of Anarchy

Mean Field Limit

Pass to the limit $N \to +\infty$?

Key assumptions:

- homogeneity: all the agents have the same f, g, b, σ
- symmetry/anonymity: interactions are only through the empirical distribution

Mean Field Limit

Pass to the limit $N \to +\infty$?

Key assumptions:

- homogeneity: all the agents have the same f, g, b, σ
- symmetry/anonymity: interactions are only through the empirical distribution

In the limit, we expect to have: the cost for one representative player is:

$$J(\alpha, \mu) = \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t, \mu_t)dt + g(X_T, \mu_T)\right]$$

with the dynamics:

$$dX_t = b(t, X_t, \alpha_t, \mu_t) + \sigma dW_t$$

where

- X and α are respectively the state and the control of the representative player,
- μ is the first marginal (state-only distribution)

Mean Field Solution Concepts

Here again, two concepts:

- **Nash equilibrium** $(\hat{\alpha}, \hat{\mu})$:
  - Optimality: for all α,
    $$J(\hat{\alpha}, \hat{\mu}) \leq J(\alpha, \hat{\mu})$$
  - Consistency: $\hat{\mu}_t = \mathcal{L}(X_t^{\hat{\alpha}})$

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- **Social optimum** α^*: for all α,
  $$J(\alpha^*, \mu^{\alpha^*}) \leq J(\alpha, \mu^{\alpha})$$
  where $\mu_t^{\alpha} = \mathcal{L}(X_t^{\alpha})$

  $\rightarrow$ no incentive for joint deviations

  $\rightarrow$ **optimization** problem for $\alpha \mapsto J(\alpha, \mu^{\alpha})$

Optimality Conditions

Large(st) part of the MFG literature focuses on equations of the form:

INTRODUCTION

This paper is devoted to the analysis of second order mean field games systems with a local coupling. The general form of these systems is:

$$\begin{cases} (i) & -\partial_t \phi - A_{ij}\partial_{ij}\phi + H(x, D\phi) = f(x, m(x,t)) \\ (ii) & \partial_t m - \partial_{ij}(A_{ij}m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \\ (iii) & m(0) = m_0, \ \phi(x,T) = \phi_T(x) \end{cases} \tag{1}$$

Source: Cardaliaguet, P., Graber, P.J., Porretta, A. and Tonon, D., 2015. Second order mean field games with degenerate diffusion and local coupling. Nonlinear Differential Equations and Applications NoDEA, 22(5), pp.1287-1317.

In a nutshell, the probabilistic approach to the solution of the mean-field game problem results in the solution of a FBSDE of the McKean–Vlasov type

$$\begin{cases} dX_t = b\big(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big)dt + \sigma dW_t, \\ dY_t = -\partial_x H\big(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big)dt + Z_t dW_t, \end{cases} \tag{3.1}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^d$, and terminal condition $Y_T = \partial_x g(X_T, \mathbb{P}_{X_T})$.

Source: Carmona, R. and Delarue, F., 2013. Probabilistic analysis of mean-field games. SIAM Journal on Control and Optimization, 51(4), pp.2705-2734.

$\rightarrow$ Theory: derivation, analysis, . . .

“Classical” Numerical Methods for MFG: Some references

Some methods based on the deterministic approach to MFG/MFC:

- Finite difference & Newton method: [ACD10], [ACCD12], ...
- (Semi-)Lagrangian approach: [CS14, CS15], [CS18], [CCS22], ...
- Augmented Lagrangian & ADMM: [BC15], [And17a], [AL16], ...
- Primal-dual algo.: [BnAKS18], [BnAKK+19], ...
- Gradient descent based methods [LP16], [Pfe16], [LP22], ...
- Monotone operators [AFG17], [GS18], [GY20], ...
- Policy iteration [CCG21a], [CK21a], [CT22], [TS22], [LST23], ...
- Finite elements [BC15], [And17b], ...
- Gaussian processes [MYZ22], ...
- Kernel-based representation [LJL+21], ...
- Fourier approximation [N+19], ...

“Classical” Numerical Methods for MFG: Some references

Some methods based on the probabilistic approach to MFG/MFC:

- Cubature [dRT15], ...
- Markov chain approximation: [BBC18], ...
- Probabilistic approach and Picard: [CCD19], [AGL+19], ...
- Probabilistic approach and regression: [BHL+19], ...
- ...

“Classical” Numerical Methods for MFG: Shortcomings

Many of these methods are very efficient and have been analyzed in detail

However, they are usually limited to problems with:

- (relatively) small dimension
- (relatively) simple structure

$\Rightarrow$ motivations to develop deep learning methods

Deep Learning Numerical Methods for MFG

- DL for direct approach for MFG [FZ20], [CL22], ...
- DL for McKean-Vlasov FBSDEs [FZ20], [CL22], [GMW22], ...
- DL for PDE system [AACN+19], [CL21], [ROL+20], [CGL20], ...
- DL for Master equations [GLPW22], [Lau21, Section 7.2], ...

Pros & Cons:

- Scalability in terms of dimension
- Much less understood than classical methods

  $\Rightarrow$ Lots of open questions for mathematicians!

Extensions

From the modeling viewpoint, many possible extensions:

- More settings, e.g. MFG with **ergodic** cost [CLLP12], [Fel13], [BP14], [ABC17b], [AKS23], ...
- Interactions through the **action distribution** ("extended MFGs", "MFGs of controls", ...): [GPV14], [GV16], [CL18], [AK20], [LT22], [Kob22], ...
- **Common noise**: in the continuous space case see [CD18] and references therein; in the finite state case, see e.g. [BLL19], [BCCD21], ...
- **Several populations** MFGs: [HMC+06b], [Fel13], [Cir15], [ABC17a], [BHL18], ...
- **Mean field type games**: [DTT17], [BGT21] and references therein; [MP19a], [CP19], [CLT19a], ...
- **Mean field control games**: [ADF+22b], [ADF+22a]

Extensions

- **Major player**: [CZ16], [CK16], [CW17], [LL18], [CCP20], [CD21], [CDL22], ...
- **Stackelberg** MFGs [BCY15], [MB18], [EMP19], [FSJ21], [ACDL22b], [VB22], [GHZ22], [DL23],...
- **Graphon** games [PO19], [CH19], [CH21], [LS22], [GTC20], [VMV21], [CCGL22], [ACL22], [ACDL22a], [BWZ23], ...
- **Correlated** equilibria [CF22], [MRE+21], [MER+22], ...
- ...

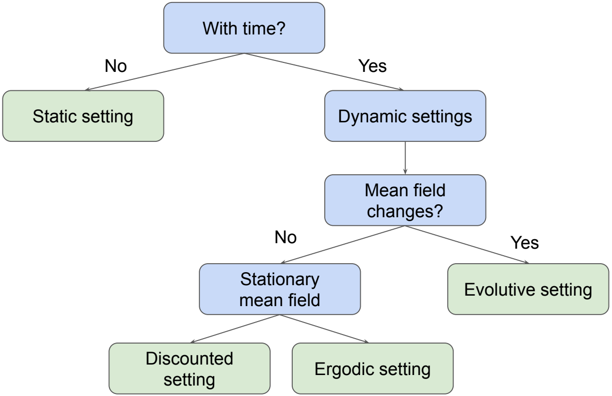
For simplicity, in most of the presentation, we will consider

- “plain” MFGs/MFCs,
- with discrete time and spaces

but many ideas can be extended in a (more or less) straightforward way.

Outline

Settings: Intuition



Settings

4 different settings:

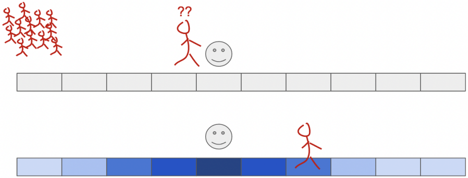
- **Static:**
  - **No states** (normal-form game): each player chooses an **action** $a \sim \pi(\cdot)$
  - Reward: depends on own action & population's action distribution
  - Examples: towel on the beach, urban settlement, ...
- **Evolutive:**
  - One-step reward: depends on own state, action & population's (state,action) distribution.
  - Fixed initial state distribution; finite or infinite time horizon.
  - Policy: **time-dependant policy** $\pi_n(\cdot|x)$
  - Examples: crowd motion, traffic routing, ...
- **Infinite horizon discounted & stationary:**
  - One-step reward: similar to Evolutive case.
  - Total reward: infinite horizon discounted sum.
  - Initial state distribution = stationary distribution induced by the population's policy.
  - Policy: **stationary policy** $\pi(\cdot|x)$
  - Examples: player joining a crowd already in a steady state
- **Ergodic:**
  - Similar to infinite horizon discounted & stationary.
  - But: Total reward = long time average.
- Other settings: asymptotic, γ-discounted, ...

Outline

Static game

Example: Population distribution (towel on the beach, . . .)

- action: choice of position
- reward: depends on my position and on the density of people



Source: unsplash

Static setting

- Finite action set A *(e.g., beach = possible towels' positions)*
- Player's behavior $\pi \in \Delta_A = \mathcal{P}(A)$
- Population's behavior $\xi \in \Delta_A$
- Player's reward: for player policy $\pi \in \Delta_A$ and population behavior $\xi \in \Delta_A$,

$$J(\pi; \xi) = \mathbb{E}_{a \sim \pi}\left[r(a, \xi)\right]$$

(e.g., crowd aversion, ice cream stall attraction, ...)

Static game: Solution concepts

- **Static MFG Nash equilibrium:** $(\hat{\pi}, \hat{\xi}) \in \Delta_A \times \Delta_A$ s.t.
  1. Best response: $\hat{\pi} \in \mathrm{BR}(\hat{\xi}) := \mathrm{argmax}_\pi J(\pi; \hat{\xi})$
  2. Consistency: $\hat{\xi} = \hat{\pi}$
- **Static MFC Social optimum:** $\pi^* \in \Delta_A$ s.t.
  - Optimality: $\pi^* \in \mathrm{argmax}_\pi J(\pi; \pi)$

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- **Static MFC Social optimum:** $\pi^* \in \Delta_A$ s.t.
  - Optimality: $\pi^* \in \mathrm{argmax}_\pi J(\pi; \pi)$
- Note: at social optimum, the population distribution is $\xi^* = \pi^*$
- But in general $\pi^* \neq \hat{\pi}$ so $\hat{\xi} \neq \xi^*$

Nash Equilibrium vs Social Optimum: Example

Consider: $A = \{1, 2\}, \qquad r(a, \xi) = c\mathbf{1}_{a=1} - \xi(a)$ where

- the constant $c \in (0, 1)$ gives some attraction to action $a = 1$
- $-\xi(a)$ is a repulsion term (crowd aversion)

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Then:

- **Static MFG Nash equilibrium:** $(\hat{\pi}, \hat{\xi}) \in \Delta_A \times \Delta_A$ s.t.
  1. Best resp.: $\hat{\pi} \in \mathrm{BR}(\hat{\xi}) := \mathrm{argmax}_\pi J(\pi; \hat{\xi}) = \pi(1)(c - \hat{\xi}(1)) + \pi(2)(-\hat{\xi}(2))$
  2. Consistency: $\hat{\xi} = \hat{\pi}$

  Is $\xi = (\xi(1), \xi(2)) = (1, 0)$ be a Nash equilibrium? Then $c - \xi(1) = c - 1 < 0 = -\xi(2)$ so $\pi = (\pi(1), \pi(2)) = (0, 1)$ would be *the* BR. Contradiction!

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  Contradiction!
  So at equilibrium both actions are optimal: $c - \hat{\xi}(1) = -\hat{\xi}(2)$
  Since $\hat{\xi}(1) + \hat{\xi}(2) = 1$, the equilibrium distrib. is: $\hat{\xi} = (\hat{\xi}(1), \hat{\xi}(2)) = (\frac{1+c}{2}, \frac{1-c}{2})$

Nash Equilibrium vs Social Optimum: Example

Consider: $A = \{1, 2\}, \qquad r(a, \xi) = c\mathbf{1}_{a=1} - \xi(a)$ where

- the constant $c \in (0, 1)$ gives some attraction to action $a = 1$
- $-\xi(a)$ is a repulsion term (crowd aversion)

Then:

- **Static MFG Nash equilibrium:** $(\hat{\pi}, \hat{\xi}) \in \Delta_A \times \Delta_A$ s.t.
  1. Best resp.: $\hat{\pi} \in \mathrm{BR}(\hat{\xi}) := \mathrm{argmax}_\pi\, J(\pi; \hat{\xi}) = \pi(1)(c - \hat{\xi}(1)) + \pi(2)(-\hat{\xi}(2))$
  2. Consistency: $\hat{\xi} = \hat{\pi}$

  Is $\xi = (\xi(1), \xi(2)) = (1, 0)$ be a Nash equilibrium? Then
  $c - \xi(1) = c - 1 < 0 = -\xi(2)$ so $\pi = (\pi(1), \pi(2)) = (0, 1)$ would be *the* BR.
  Contradiction!
  So at equilibrium both actions are optimal: $c - \hat{\xi}(1) = -\hat{\xi}(2)$
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  and since $\pi(2) = 1 - \pi(1)$, $J(\pi; \pi) = -1 + (2 + c)\pi(1) - 2\pi(1)^2$ First order optimality condition gives:
  $0 = \frac{d}{d\pi(1)}[-1 + (2 + c)\pi^*(1) - 2\pi^*(1)^2] = (2 + c) - 4\pi^*(1)$
  so the socially optimum distribution is: $\xi^* = (\xi^*(1), \xi^*(2)) = (\frac{2+c}{4}, \frac{2-c}{4})$

Nash Equilibrium vs Social Optimum: Example

Consider: $A = \{1, 2\}, \qquad r(a, \xi) = c\mathbf{1}_{a=1} - \xi(a)$ where

- the constant $c \in (0, 1)$ gives some attraction to action $a = 1$
- $-\xi(a)$ is a repulsion term (crowd aversion)

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- So, in this example with $c \in (0, 1)$, $\hat{\xi} \neq \xi^*$
- Nash equilibrium is more concentrated on action 1 than MFC (“selfishness”)

Nash Equilibrium vs Social Optimum: Potential case

- In *some cases*, the two notions coincide.
- Example: Potential MFG with reward: $r(a, \xi) = \nabla F(\xi)(a)$ for some $F : \Delta_A \to \mathbb{R}$
- The average cost is: $J(\pi, \xi) = \mathbb{E}_{a\sim\pi}[r(a, \xi)] = \sum_a \pi(a)\nabla F(\xi)(a) = \pi \cdot \nabla F(\xi)$

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- Assuming the potential F concave, we have the equivalence:

$$
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\hat{\pi} \text{ is a NE} &\Leftrightarrow J(\pi,\hat{\pi}) - J(\hat{\pi},\hat{\pi}) \leq 0, \quad \forall \pi \\
&\Leftrightarrow (\pi - \hat{\pi}) \cdot \nabla F(\hat{\pi}) \leq 0, \quad \forall \pi \\
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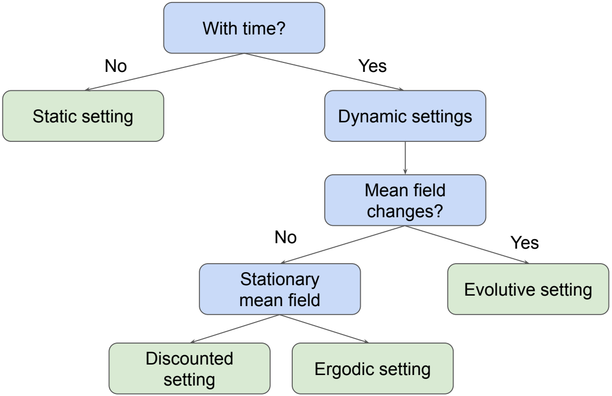
- Example: (negative of) entropy: $F(\xi) = -\sum_a \xi(a) \log(\xi(a))$: encourages agent to spread throughout the action space A
- Note: the link between potential MFGs and MFC can be exploited to design numerical methods

Settings: Intuition – Reminder



Outline

Notation for Dynamic Settings

- State $x \in S$, action $a \in A$ (S, A finite for most of this presentation)
- Mean field state $\mu \in \Delta_S = \mathcal{P}(S)$ (extensions: state-action distrib.)
- Discrete time $n \in \mathbb{N}$
- Player's transition probability: $p(\cdot|x, a, \mu)$
- Player's reward: $r(x, a, \mu)$
- One-step policy: $\pi \in \Pi := (\Delta_A)^S$, functions $S \to \Delta_A$
- One-step mean field transition matrix: $P_{\mu,\pi}(x, y) = \sum_{a \in A} \pi(a|x) p(y|x, a, \mu)$

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- One-step mean field transition matrix: $P_{\mu,\pi}(x, y) = \sum_{a \in A} \pi(a|x) p(y|x, a, \mu)$
- What happens in one time step?
  - "Each" player selects an action (we focus on one "representative" player)
  - "Each" player gets a reward
  - "Each" player state is updated
  - Mean field is updated
- Mathematically: with policy π_n and mean field μ_n

$$a_n \sim \pi_n(\cdot|x_n)$$

$$r(x_n, a_n, \mu_n)$$

$$x_{n+1} \sim p(\cdot|x_n, a_n, \mu_n)$$

$$\mu_{n+1} = P^\top_{\mu_n,\pi_n} \mu_n = \sum_{y \in S} \mu_n(y) \sum_{a \in A} \pi_n(a|y) p(\cdot|y, a, \mu_n)$$

Stationary setting

Stationary game

Example: joining a population in a stationary regime (flocking, economics, ...)

- the population is at equilibrium $\rightarrow$ MF distribution is stationary
- a player wants to join $\rightarrow$ optimal control problem
- but the distribution is the result of the agents' decisions $\rightarrow$ fixed point problem

Source: unsplash

Stationary setting

- Stationary setting: $N_T = \infty$
- No fixed initial m_0 but a stationary distribution
- Notation: $\mathrm{MF}(\pi) :=$ stationary distribution when using policy π:

$$\mu = P^{\top}_{\mu,\pi}\mu =: \mathcal{P}^{\pi}(\mu)$$

- Player's reward: for player's policy $\pi \in \Delta_A$ and mean field $\mu \in \Delta_S$,

$$J(\pi; \mu) = \mathbb{E}\left[\sum_{n=0}^{\infty} \gamma^n r(x_n, a_n, \mu)\right]$$

where $\gamma \in (0, 1)$ is a discount parameter, and

$$a_n \sim \pi(\cdot|x_n), \qquad x_0 \sim \mu, \quad x_{n+1} \sim p(\cdot|x_n, a_n, \mu), n \geq 0$$

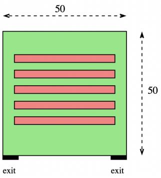
Stationary setting: Solution concepts

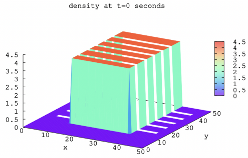
- **Stationary MFG Nash equilibrium:** $(\hat{\pi}, \hat{\mu}) \in \Pi \times \Delta_{S \times A}$ s.t.
  1. Best response: $\hat{\pi} \in \mathrm{BR}(\hat{\mu}) := \mathrm{argmax}_{\pi} J(\pi; \hat{\mu})$
  2. Mean field state: $\hat{\mu} = \mathrm{MF}(\hat{\pi})$
- Fixed point: $\hat{\mu} \in \mathrm{MF}(\mathrm{BR}(\hat{\mu}))$

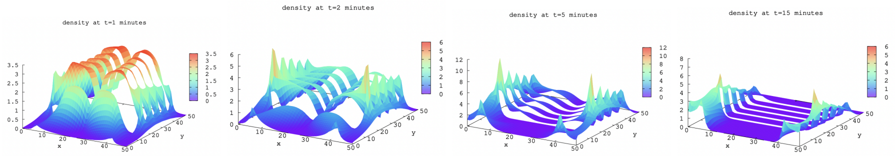
- **Stationary MFC Social optimum:** $\pi^* \in \Pi$ s.t.
  - Optimality: $\pi^* \in \mathrm{argmax}_{\pi^*} J(\pi^*; \mu^{\pi^*})$ where $\mu^{\pi^*} = \mathrm{MF}(\pi^*)$

Evolutive setting

Example: Crowd exiting a room [AL15]







Evolutive setting

- Horizon: $N_T \in \mathbb{N}$ (extensions: p, r depending on n; infinite horizon)
- Fixed initial state distribution: $m_0 \in \Delta_S$
- The MF evolves in time: $\boldsymbol{\mu} = (\boldsymbol{\mu}_n)_{n=0,\dots,N_T} \in \Delta_S^{N_T}$
- Notation $\mathrm{MF}_{m_0,N_T}(\boldsymbol{\pi}) :=$ generated by policy $\boldsymbol{\pi}$ starting from m_0:

$$\begin{cases} \boldsymbol{\mu}_0 = m_0, \\ \boldsymbol{\mu}_{n+1} = P^\top_{\boldsymbol{\mu}_n, \boldsymbol{\pi}_n} \boldsymbol{\mu}_n, \quad n \geq 0 \end{cases}$$

- Player's reward: for player's policy $\boldsymbol{\pi} \in \Pi^{N_T}$ and mean field $\boldsymbol{\mu} \in \Delta_S^{N_T}$,

$$J(\boldsymbol{\pi}; \boldsymbol{\mu}) = \mathbb{E}\left[\sum_{n=0}^{N_T} r(x_n, a_n, \boldsymbol{\mu}_n)\right]$$

where

$$a_n \sim \boldsymbol{\pi}_n(\cdot|x_n), \qquad x_0 \sim m_0, \quad x_{n+1} \sim p(\cdot|x_n, a_n, \boldsymbol{\mu}_n), n \geq 0$$

Evolutive setting: Solution concepts

- **Evolutive MFG Nash equilibrium:** $(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\mu}}) \in \Pi^{N_T} \times \Delta_S^{N_T}$ s.t.
  1. Best response: $\hat{\boldsymbol{\pi}} \in \mathrm{BR}(\hat{\boldsymbol{\mu}}) := \mathrm{argmax}_{\boldsymbol{\pi}} J(\boldsymbol{\pi}; \hat{\boldsymbol{\mu}})$
  2. Mean field flow: $\hat{\boldsymbol{\mu}} = \mathrm{MF}_{m_0, N_T}(\hat{\boldsymbol{\pi}})$
- Fixed point: $\hat{\boldsymbol{\mu}} \in \mathrm{MF}_{m_0, N_T}(\mathrm{BR}(\hat{\boldsymbol{\mu}}))$

- **Evolutive MFC Social optimum:** $\boldsymbol{\pi}^* \in \Pi^{N_T}$ s.t.
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Outline

Value function: stationary case

- Value function of a (stationary) policy π given a (stationary) mean field μ: $V^{\mu,\pi}(x) := \mathbb{E}_\pi\left[\sum_{n\geq 0}\gamma^n r(x_n, a_n, \mu)\right]$ satisfies:

$$V^{\mu,\pi}(x) = \mathbb{E}_{a\sim\pi(\cdot|x)}\Big[\underbrace{r(x,a,\mu) + \gamma\mathbb{E}_{x'\sim p(\cdot|x,a,\mu)}[V^{\mu,\pi}(x')]}_{Q^{\mu,\pi}(x,a)}\Big]$$

$$V^{\mu,\pi} = \mathcal{T}^{\mu,\pi}V^{\mu,\pi}$$
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- Optimal value function given a mean field μ: $V^{\mu,*}(x) = \max_\pi V^{\mu,\pi}(x)$:

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- Optimal policy given a mean field μ: single player's problem:

$$\mathrm{supp}(\pi^*(\cdot|x)) \subseteq \operatorname*{argmax}_{a\in A} Q^{\mu,*}(x,a)$$

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Value function: finite horizon evolutive case

Finite horizon evolutive case ($N_T < +\infty$):

- Value function of a policy $\boldsymbol{\pi}$ given a mean field $\boldsymbol{\mu}$: $V_n^{\boldsymbol{\mu},\boldsymbol{\pi}}(x) := \mathbb{E}_{\boldsymbol{\pi}}[\sum_{n'=n}^{N_T} r(x_{n'}, a_{n'}, \boldsymbol{\mu}_{n'}) | x_n = x]$ satisfies:

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- Optimal value function given a mean field $\boldsymbol{\mu}$:

$$V_n^{\boldsymbol{\mu},*}(x) = \max_{\boldsymbol{\pi}} V_n^{\boldsymbol{\mu},\boldsymbol{\pi}}(x)$$

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- Bellman equations are **backward induction equations**

Outline

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We are going to focus mostly on MFG Nash equilibria computation

MFG Equilibrium Computation: General Principles

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Two main objects: policy π and population distribution μ

Most basic idea: alternate

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Many other possibilities using optimality conditions, e.g.

- traditional methods such as Newton's method for the PDE system [ACCD12]
- deep learning methods for PDE/FBSDE system, see [HL22]

But cannot be directly adapted to the model-free RL setting.

Updating the policy

For standard MDPs:

- Bellman operators
  - Optimal Bellman operator:

$$\mathcal{B}^* : (Q(x,a))_{x,a} \mapsto \mathcal{B}^* Q = \left(r(x,a) + \gamma \mathbb{E}_{x' \sim p(\cdot|x,a)} [\max_{a'} Q(x', a')] \right)_{x,a}$$

  - Bellman operator associated to a policy π:

$$\mathcal{B}^\pi : (Q(x,a))_{x,a} \mapsto \mathcal{B}^\pi Q = \left(r(x,a) + \gamma \mathbb{E}_{x' \sim p(\cdot|x,a), a' \sim \pi} [Q(x, a')] \right)_{x,a}$$

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  - **Value iteration:**

$$Q^{k+1} = \mathcal{B}^* Q^k$$

  - **Policy iteration:**

$$\begin{cases} Q^{k+1} = Q^{\pi^k} & \text{(policy evaluation)} \\ \pi^{k+1} \in \operatorname{argmax} Q^{k+1} & \text{(policy improvement)} \end{cases}$$

where the policy evaluation can be done by applying $\mathcal{B}^{\pi^k}$ many times

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$\to$ For MFG: intertwine applications of $\mathcal{B}^{\mu,*}$ or $\mathcal{B}^{\mu,\pi}$ with MF updates

Iterative methods for MFG: Stationary case

Goal: find MFG Nash equilibirum $(\hat{\pi}, \hat{\mu}) \in \Pi \times \Delta_S$

- Iterations based on **Best response computation**:
  1. Compute best response: $\pi^{k+1} = \mathrm{BR}(\mu^k)$:
     - 1.1 Compute the optimal value function: $Q^{\mu^k,*} = \mathcal{B}^{\mu^k,*} Q^{\mu^k,*}$
     - 1.2 Let: $\pi^{k+1}(\cdot|x) \in \mathrm{argmax}_a\, Q^{\mu^k,*}(x,a)$
  2. Compute stationary MF: $\mu^{k+1} = \mathrm{MF}(\pi^{k+1})$: $\mu^{k+1} = \mathcal{P}^{\pi^{k+1}} \mu^{k+1}$
- Iterations based on **Policy evaluation** ("policy iteration"):
  1. Update policy:
     - 1.1 Evaluate policy: $Q^{\mu^k,\pi^k} = \mathcal{B}^{\mu^k,\pi^k} Q^{\mu^k,\pi^k}$
     - 1.2 Let: $\pi^{k+1}(\cdot|x) \in \mathrm{argmax}_a\, Q^{\mu^k,\pi^k}(x,a)$
  2. Compute stationary MF: $\mu^{k+1} = \mathrm{MF}(\pi^{k+1})$: $\mu^{k+1} = \mathcal{P}^{\pi^{k+1}}(\mu^{k+1})$

Iterative methods for MFG: Stationary case

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Sometimes: one application of fixed point operator instead of true fixed point:

- $\mu^{k+1} = \mathcal{P}^{\pi^{k+1}}(\mu^k)$ instead of μ^{k+1} s.t. $\mu^{k+1} = \mathcal{P}^{\pi^{k+1}}(\mu^{k+1})$
- Learning step $\approx$ time step in the game

Iterative methods for MFG: Evolutive case

Goal: find MFG Nash equilibrium $(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\mu}}) \in \Pi^{N_T} \times \Delta_S^{N_T}$

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     - 1.2 Let: $\boldsymbol{\pi}_n^{k+1}(\cdot|x) \in \mathrm{argmax}_a\, Q_n^{\boldsymbol{\mu}^k,*}(x,a)$
  2. Compute MF flow: $\boldsymbol{\mu}^{k+1} = \mathrm{MF}_{m_0,N_T}(\boldsymbol{\pi}^{k+1})$
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  1. Update policy:
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Backward equations instead of fixed point equations as in stationary case

Challenges

Potential issues (for both stationary and evolutive settings):

- Non-uniqueness of the equilibrium MF $\hat{\mu}$ or $\hat{\boldsymbol{\mu}}$

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  $\Rightarrow$ Oscillations / instabilities

Several variations / improvements have been studied

Outline

Damping / Averaging

Damping / smoothing:

- for policies: instead of:
  $$\pi^{k+1} = \mathrm{BR}(\mu^k)$$
  use:
  $$\bar{\pi}^{k+1} = \sum_{i=1}^{k} \alpha_i \mathrm{BR}(\mu^i)$$
  for some coefficients $(\alpha_i)_i$, and then:
  $$\mu^{k+1} = \mathrm{MF}(\bar{\pi}^{k+1})$$
- and/or average mean fields, value functions, ...

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$\rightarrow$ Encompasses many possible variants such as:

- Fixed point iteration / value iteration (no damping): e.g. [HMC06a, GHXZ19, AKS20b] ...
- Fictitious Play: e.g. [CH17, Had17, MJMdC18, PPL+20, MH21, DV21] ...
- Policy Iteration: e.g. [CCG21b, CT21, LST21] ...
- Online Mirror Descent (OMD): e.g. [Had17, Had18, PPE+21a] ...

Smooth policies

Class of smooth(er) policies:

- E.g. softmax/Botzmann policies: instead of

$$\pi^{k+1}(\cdot|x) \in \operatorname{argmax} Q^k(x,\cdot)$$

use:

$$\pi^{k+1}(\cdot|x) = \text{softmax}_\tau Q^k(x,\cdot) = \frac{e^{\frac{1}{\tau}Q(x,\cdot)}}{\sum_a e^{\frac{1}{\tau}Q(x,a)}}$$

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- forces to play every action with a positive probability
- temperature τ can be decreased progressively if needed
- solves the problem of ambiguity among possible elements of argmax
- but the equilibrium policy $\hat{\pi}$ is not necessarily of $\operatorname{softmax}$ form!

Reward regularization

Reward regularization:

- Modify the reward with a regularizing penalty
- For instance, entropy penalty: instead of:

$$r(x, a, \mu)$$

use:

$$r(x, a, \mu) - \eta \log \left(\frac{\pi(a|x)}{\tilde{\pi}(a|x)} \right)$$

where $\tilde{\pi}$ is a reference policy (e.g., uniform)

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  where $\tilde{\pi}$ is a reference policy (e.g., uniform)
- it depends on the whole policy $\pi(\cdot|x)$ and not just on the action played
- helps to ensure uniqueness of the equilibrium and the BR
- but only for the *modified* game $\neq$ original game

Some Canonical Examples

Algorithm: Fixed point iter.

input : Initial policy π^0

$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$ **do**
  $\pi^k :=$ BR against μ^{k-1};
  $\mu^k := \mu^{\pi^k}$;
return π^K, μ^K

$\downarrow$

Algorithm: Fictitious Play

input : Initial policy π^0

$\bar{\pi}^0 := \pi^0$;
$\bar{\mu}^0 := \mu^{\bar{\pi}^0}$;
for $k = 1, \dots, K$ **do**
  $\pi^k :=$ BR against $\bar{\mu}^{k-1}$;
  $\bar{\mu}^k := \frac{k}{k+1}\bar{\mu}^{k-1} + \frac{1}{k+1}\mu^{\pi^k}$;
  $\bar{\pi}^k :=$ policy giving $\bar{\mu}^k$;
return $\bar{\pi}^K$, $\bar{\mu}^K$

Algorithm: Policy iter.

input : Initial policy π^0

$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$ **do**
  $Q^k :=$ Q-func. for π^{k-1} given μ^{k-1};
  $\pi^k := \mathrm{argmax}\ Q^k$;
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return π^K, μ^K

$\downarrow$

Algorithm: OMD

input : Initial policy π^0

$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$ **do**
  $Q^k :=$ Q-func. for π^{k-1} given μ^{k-1};
  $\bar{Q}^k := \bar{Q}^{k-1} + \alpha Q^k$;
  $\pi^k := \mathrm{softmax}_\tau\ \bar{Q}^k$;
  $\mu^k := \mu^{\pi^k}$;
return π^K, μ^K

Assumptions and convergence guarantees

Several classes of assumptions to guarantee convergence of the iterations:

1. "**Quantitative**" assumptions:
   - small Lipschitz constants / short time
   - proof by strict contraction
   - Ex: [HMC06a, GHXZ19, AKS20b, LST21] . . .

2. "**Qualitative**/structural" assumptions:
   - potential structure / monotonicity
   - proof by Lyapunov stability
   - Ex: [CH17, Had17, Had18, MJMdC18, PPL+20, PPE+21a] . . .

Convergence?

How can we check whether the algorithm has converged?

Beware:

- Total reward of a player is not a good indicator of convergence
- Distance between π and $\hat{\pi}$ is not necessarily meaningful

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$\rightarrow$ **Exploitability:**

- Evaluates the quality of a policy in a game [ZJBP07, LWZB09]
- *How "far" π is from being a Nash equilibrium policy?*

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In the context of MFGs:

- Definition: The **exploitability** $\mathcal{E}(\pi)$ of a policy π is defined as:

$$\mathcal{E}(\pi) := \max_{\pi'} J(\pi', \mu^{\pi}) - J(\pi, \mu^{\pi})$$

- Interpretation: $\mathcal{E}(\pi)$ quantifies the average gain for a representative player to replace its policy by a best response, while the rest of the population plays with policy π.
- If $\mathcal{E}(\pi) = 0$, then π is a Nash equilibrium policy.

Outline

OpenSpiel

- Open source framework for research in learning in games
- Main motivation: multi-agent reinforcement learning (MARL)
- Marc Lanctot (Google DeepMind) + many contributors
- Mostly in C++ and Python; APIs in Julia, ...
- Various games including zero-sum games, N-player games, imperfect information, ...
- Chess, Blackjack, Atari, Kuhn poker, Go, ...
- And also: Mean field games

OpenSpiel

Introduction to OpenSpiel:

- https://openspiel.readthedocs.io/en/latest/intro.html
- Python notebook: https://colab.research.google.com/github/deepmind/open_spiel/blob/master/open_spiel/colabs/OpenSpielTutorial.ipynb
- Tutorial by Marc Lanctot available online: https://www.youtube.com/watch?v=8NCPqtPwlFQ
- Paper [LLL+19]
- Two big components:
  - Games
  - Algorithms

MFG in OpenSpiel

- Julien Pérolat, Raphael Marinier, Sertan Girgin & growing number of contributors Théophille Cabannes, Sarah Perrin, Paul Muller, . . .
- For today, three main questions:
  - How to use the existing material?
  - How to define a new MFG model (environment/game)?
  - How to define a new algorithm to learn the MFG solution?

Existing codes for MFG in OpenSpiel

- MFG models in C++: `https://github.com/deepmind/open_spiel/tree/master/open_spiel/games/mfg`
- MFG models in Python: `https://github.com/deepmind/open_spiel/tree/master/open_spiel/python/mfg/games`
  - Crowd modeling 1D illustrated in [PPL+20]
  - Crowd modeling 2D illustrated in [PPL+20, GPL+22]
  - Dynamic routing illustrated in [CLP+22]
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  - Deep fictitious play [LPG+22]
  - Boltzmann policy iteration [CK21a]
  - Fictitious play [PPL+20], ...
  - Fixed point
  - Mirror descent [PPE+21b]
  - Munchausen deep mirror descent [LPG+22]
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  as well as codes for policies and an evaluation metric: exploitability (nash_conv)

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  as well as codes for policies and an evaluation metric: exploitability (nash_conv)
- Some examples: `https://github.com/deepmind/open_spiel/tree/master/open_spiel/python/mfg/examples`

More to come soon. Contributions are welcome!

Q1. *How to use existing material?*

- Install & imports
- Creating a game (e.g., grid world)
- Running a learning algorithm (e.g., fictitious play)
- Plotting the results (e.g., exploitability and distribution)

Tutorial 1: Introduction to MFG in OpenSpiel

Code

Sample code to illustrate: IPython notebook

https://colab.research.google.com/drive/16p95oXZGdhzCAX9MTPlcMNnsD3dyW9ur?usp=sharing

- Installation and imports
- Creating a game
- Running an algorithm
- Visualizing the results

* Special thanks to Marc Lanctot, Julien Pérolat, Raphael Marinier, Sertan Girgin, Sarah Perrin and Kai Shao for this notebook

Tutorial 2: Comparing Learning Algorithms

Another example of game: 2D crowd modeling in a grid world but with obstacles (4 connected rooms). The performance of several algorithms are compared.

Code

Sample code to illustrate: IPython notebook

https://colab.research.google.com/drive/1L1MIVba_2Wm534TDcGL35W2D5vxCsFeo?usp=sharing

- Four room grid world
- Running multiple pre-defined algorithms
- Comparing their exploitabilities

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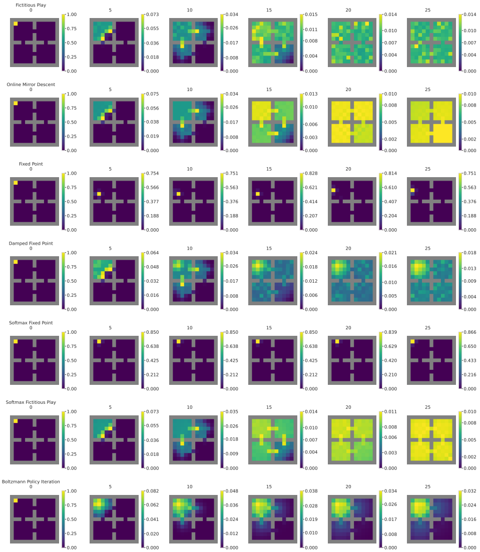
Comparing Learning Algorithms – Results

Game: crowd aversion in a four-room grid world

Test case 1: Noise level = 0.2

State distribution at different time steps (columns) for different algorithms (rows):



Comparing Learning Algorithms – Results

Game: crowd aversion in a four-room grid world
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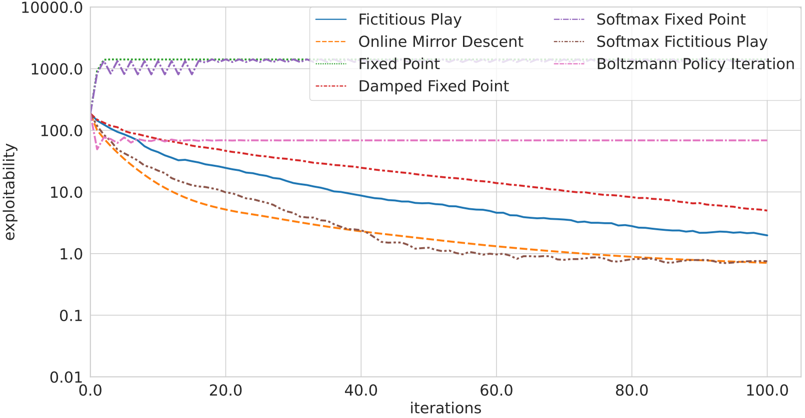
Exploitability vs number of steps:



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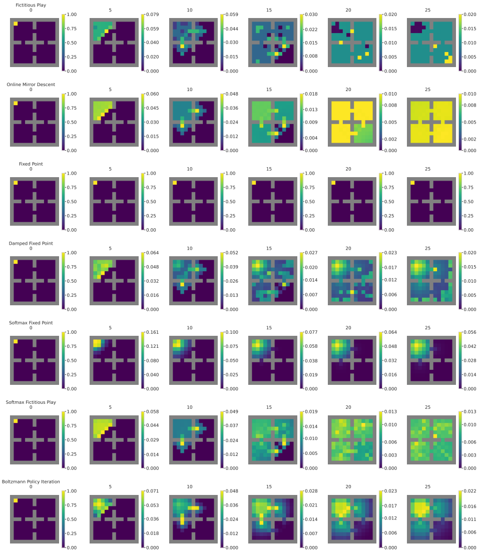

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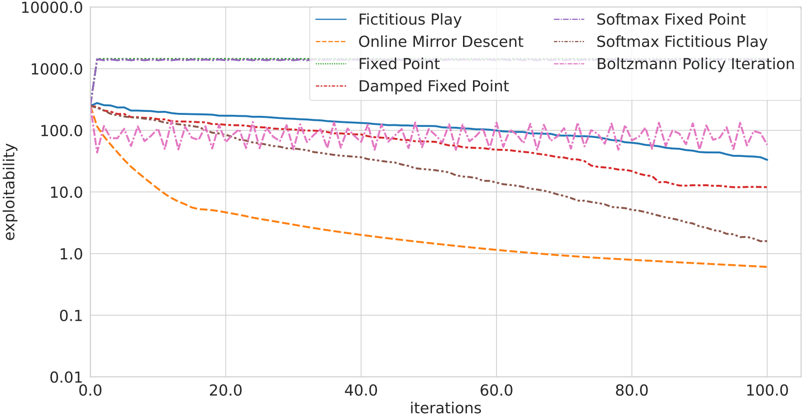

MFG model in OpenSpiel: State

Q2. *How to define a new MFG model?*

- State of the game = all the information required to describe the current stage
- In an MFG: representative player's state and mean field state
- Evolution of the state:
  - Players play in turn
  - Every change to the state occurs through a node
  - Each node has a set of possible actions and a probability to pick each action

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  - and the "noise" is viewed as a node too

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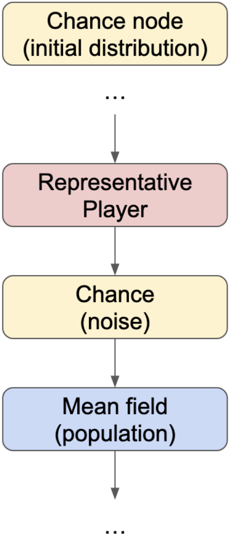
  - and the "noise" is viewed as a node too
  - Time is part of the state: (t, x)
- The state evolves along a tree of possibilities

MFG model in OpenSpiel: State evolution



MFG model in OpenSpiel: State types

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  - probabilities: given by the initial state distribution

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  - actions: set of possible values for the noise impacting the dynamics
  - probabilities: distribution of the noise values
- **Mean field:** no actions

MFG in OpenSpiel: Distribution

- The distribution is something specific to MFGs (compared with other games in OpenSpiel)
- Remember that time is part of the state object. Evaluating the distribution at a given state means evaluating the distribution at (t, x).
- `master/open_spiel/python/mfg/algorithms/distribution.py`
  - Computes the distribution of a policy
  - `DistributionPolicy`
    - `evaluate`: based on the logic behind nodes
    - `_one_forward_step`
- `master/open_spiel/python/mfg/distribution.py`
  - Representation of a distribution for a game
  - `Distribution`
- `master/open_spiel/python/mfg/tabular_distribution.py`
  - Tabular representation of a distribution for a game
  - `TabularDistribution`

MFG model in OpenSpiel: Example

We take a concrete example: crowd modeling in 1D with a grid world

`master/open_spiel/python/mfg/games/crowd_modelling.py`

3 main classes

- `MFGCrowdModellingGame`:
  - `__init__`: initialization
  - `new_initial_state`: generate new initial state
- `MFGCrowdModellingState`:
  - `__init__`: initialization
  - `_legal_actions`: actions that are valid
  - `chance_outcomes`: distribution over values of the noise in the dynamics
  - `_apply_action`: will be called at each node to modify the state based on the action
  - `_rewards`: representative player's reward
- `Observer`:
  - defines an observation, here basically t and x

Q3. *How to define a new algorithm?*

Simplest one: Fixed point
`master/open_spiel/python/mfg/algorithms/fixed_point.py`

A bit more involved: Fictitious play
`master/open_spiel/python/mfg/algorithms/fictitious_play.py`

- Main class `FictitiousPlay`
- Main method `iteration`
  - Compute the distribution (sequence) associated to the current policy
  - Update the policy (using fictitious play rule); this uses an auxiliary class `MergedPolicy` to mix the previous policy and the new one
- `get_policy`: returns the current policy

Tutorial 3: Game construction

Code

Sample code to illustrate: IPython notebook

https://colab.research.google.com/drive/1uIcDYxQ9f7ngqIOo7ittZ4jEmXFOOZs9?usp=sharing

- Details of the definition of an MFG game in OpenSpiel
- Modification of an existing game
- Reward function, transitions, . . .

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Outline

Outline

Revisiting Dynamic Programming: Classical Setup

Classical MDP (S, A, p, r, γ):

$$Q^\pi(x,a) = (\mathcal{B}^\pi Q^\pi)(x,a) = r(x,a) + \gamma \mathbb{E}_{\substack{x' \sim p(\cdot|x,a), \\ a' \sim \pi(\cdot|x)}} \Big[Q^\pi(x',a') \Big]$$

$\rightarrow$ Can be computed by applying repeatedly $\mathcal{B}^\pi$

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$\rightarrow$ *But what if* p *&* r *are unknown and we can only observe samples* $(x', r(x,a))$*?*

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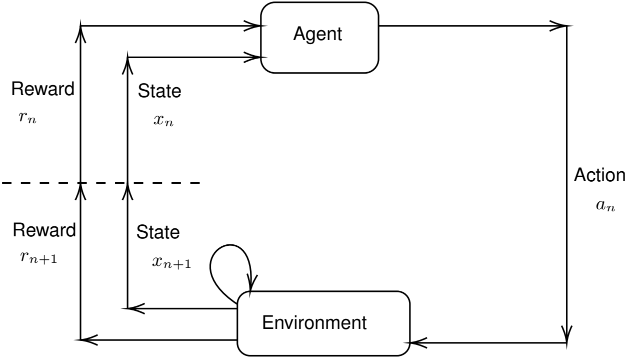

See e.g. [SB18]

Revisiting Dynamic Programming: Mean Field Game Setup

MDP parameterized by mean field term $(S, A, p(\cdot|\cdot,\cdot,\mu), r(\cdot|\cdot,\cdot,\mu), \gamma)$:

$$Q^{\mu,\pi}(x,a) = (\mathcal{B}^{\mu,\pi} Q^{\mu,\pi})(x,a) = r(x,a,\mu) + \gamma \mathbb{E}_{\substack{x' \sim p(\cdot|x,a,\mu), \\ a' \sim \pi(\cdot|x)}} \left[Q^{\mu,\pi}(x',a') \right]$$

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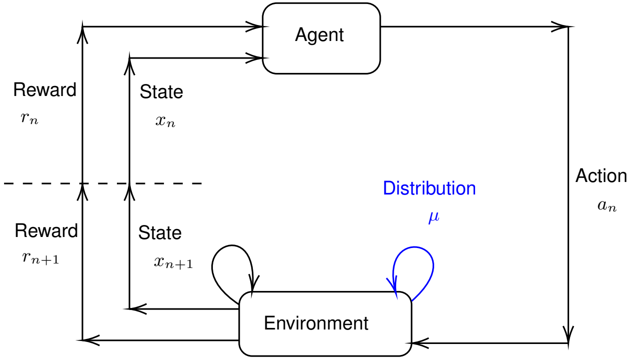

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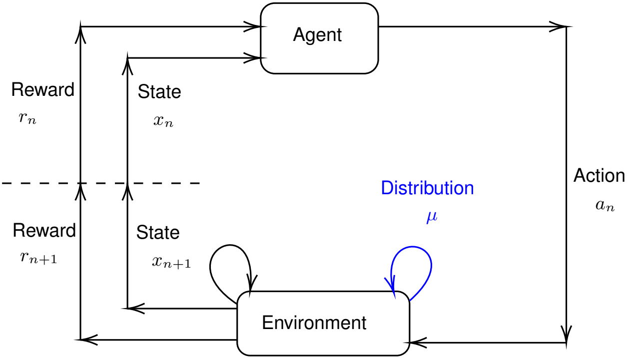

Note: the agent does not need to observe μ, but it is part of the environment.

Revisiting Dynamic Programming: Mean Field Game Setup

How to deal with μ in practice? To implement the simulator, we can for instance:

- Vector (if finite S); updates using transition matrix
- Empirical distribution μ^N; updates using individual transitions
- Neural network (e.g., normalizing flow); updates by training
- ...

Outline

Policy evaluation

Policy evaluation: given μ, π, evaluate

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Assume we can compute the expectation perfectly.

Repeatedly improve estimate Q^k of $Q^{\mu,\pi}$:

- With tabular representation: pointwise update for (x,a)

$$Q^{k+1}(x,a) = r(x,a,\mu) + \gamma \mathbb{E}_{\substack{x' \sim p(\cdot|x,a,\mu), \\ a' \sim \pi(\cdot|x)}} \left[Q^{k}(x',a') \right]$$

- With function approximation: Q^{k+1} parameterized by θ^{k+1} minimizing

$$\mathbb{E}\left[\left| Q_{\theta^{k+1}}(x,a) - r(x,a,\mu) - \gamma \mathbb{E}_{\substack{x' \sim p(\cdot|x,a,\mu), \\ a' \sim \pi(\cdot|x)}} \left[Q_{\theta^k}(x',a') \right] \right|^2 \right]$$

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Assume only samples $x' \sim p(\cdot|x,a,\mu), r(x,a,\mu)$ from the environment.

Repeatedly improve estimate Q^k of $Q^{\mu,\pi}$:

- Observe $x' \sim p(\cdot|x,a,\mu), r(x,a,\mu)$ from the environment
- Approximate $\mathbb{E}_{\substack{x' \sim p(\cdot|x,a,\mu), \\ a' \sim \pi(\cdot|x)}} \left[Q^{\mu,\pi}(x',a') \right]$ by Monte Carlo
- Use similar updates as before (in the ideal case)? For instance with tabular representation: at a given k, for all (x,a) compute:

$$Q^{k+1}(x,a) = r(x,a,\mu) + \gamma \tilde{\mathbb{E}}^B_{\substack{x' \sim p(\cdot|x,a,\mu), \\ a' \sim \pi(\cdot|x)}} \left[Q^{k}(x',a') \right]$$

  where $\tilde{\mathbb{E}}^B$ is an empirical expectation based on a batch of B i.i.d samples.
- This is model-free (= purely based on samples from the environment) ...

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- This is model-free (= purely based on samples from the environment) . . .
- But this requires: *many* samples for *every* (x,a) at *every* iteration k . . .

Policy evaluation: Model-free

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- asynchronous updates: follow a trajectory $(x^k, a^k)_{k \geq 0}$:

$$Q^{k+1}(x^k, a^k) = r(x^k, a^k, \mu) + \gamma Q^k(x^{k+1}, a^{k+1})$$

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- learning rate:

$$Q^{k+1}(x^k,a^k) = (1-\alpha) Q^k(x^k,a^k) + \alpha \Big[r(x^k,a^k,\mu) + \gamma Q^k(x^{k+1},a^{k+1}) \Big]$$

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- many extra tricks (replay buffer, policy parameterization, ...)

Optimal policy computation

Best response computation: given μ, compute

$$Q^{\mu,*}(x,a) = r(x,a,\mu) + \gamma \mathbb{E}_{x' \sim p(\cdot|x,a,\mu)}\left[\max_{a'} Q^{\mu,*}(x',a')\right]$$

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- similar as evaluation, using MC samples
- computation of $\max$ (and argmax to recover an optimal policy) possible by exhaustive search if the action space A is finite *and small*
- tabular Q-learning [WD92] (with extra μ in the environment):

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  where $x^{k+1} \sim p(\cdot|x^k,a^k,\mu)$ and $a^k \sim$ some policy
- otherwise: learn an optimal parameterized policy
  - either along the way, with the Q-function $\Rightarrow$ actor-critic methods
  - only the parameterized policy $\Rightarrow$ policy gradient methods
- Ex: DQN, SAC, PPO, . . .

MFG Nash equilibrium: Model-free

- Above: enables the computation of a Best Response
  (using for instance model-free versions of value iteration and policy iteration)

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- “Outer loop” to **update the mean field μ**
- Various options, depending on the MFG setting:
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  - Evolutive setting: application of the transition matrix for each of the time steps (computation of the MF sequence)

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- If applying the transition matrix is not an option (e.g., continuous spaces), one can for instance use an empirical distribution obtained by simulating N agents

MFG algorithms in OpenSpiel: Reinforcement Learning – Principles

OpenSpiel also contains RL codes for MFGs

Two main building blocks:

- **Environment** (in the sense of RL): in charge of updating the State based on the based on the Game
- **Agent**: in charge of training the policy by interacting with the environment

MFG algorithms in OpenSpiel: Reinforcement Learning – Examples

Policy update: best respond computation for instance through DQN:

- DQN is a variant of Q-learning with a neural network for Q [MKS+15]
- Implementation: `open_spiel/python/mfg/examples/mfg_dqn_jax.py`
- neural network implementation through JAX
- see the source code for details (hyperparameters etc.)

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Mean field update: Example of DQN embedded in Fictitious Play [LPG+22]:

- Train a NN for the *average* policy across iterations
- Implem.: `open_spiel/python/mfg/examples/mfg_dqn_fp_jax.py`
- Key steps:
  - `fp.iteration(br_policy=joint_avg_policy)`: performs one iteration of fictitious play (updates the policy and the distribution)
  - `distrib = distribution.DistributionPolicy(game, fp.get_policy())`: get the distribution induced by the new policy, just computed by fictitious play iteration
  - `env.update_mfg_distribution(distrib)`: update the environment's distribution using the one obtained from the fictitious play iteration
  - `agents[p].step(time_step)`: train the agent

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Alternative: Munchausen Deep Mirror Descent [LPG+22]:

- Train a NN for the cumulative Q-function
- Implem.: `open_spiel/python/mfg/examples/munchausen_deep_mirror_descent.py`

Tutorial 4: Deep RL for MFG in OpenSpiel

Code

Sample code to illustrate: IPython notebook

`https://colab.research.google.com/drive/1rF9DpjO_xTpbBC2Y-6h_7yQ80j75eb6j?usp=sharing`

- Installation and imports for DRL in OpenSpiel
- Munchausen Deep Mirror Descent
- Average Network Fictitious Play

* Special thanks to Marc Lanctot, Julien Pérolat, Raphael Marinier, Sertan Girgin, Sarah Perrin and Kai Shao for this notebook

A (Non-exhaustive) Glance at the literature: RL for MFG

RL for Mean Field Game:

- MARL with mean field approximation: Yang et al. [YLL+18]
- Inverse RL: Yang et al. [YYT+17], Chen et al. [CLK21]
- Multi-time scales: Subramanian et al. [SM19], Angiuli et al. [AFL20, AFLZ20, AH21]
- Fictitious Play with tabular RL: Pérolat et al. [PPL+20], with deep RL: Elie et al. [EPL+20, CK21b] and distribution embedding: Perrin et al. [PLP+21b]
- Fixed point iterations with Q-learning and variants: Guo et al. [GHXZ19, GHXZ20], Anahtarci et al. [AKS19, AKS21], Xie et al. [XYWM21]
- Entropy regularization: Anahtarci et al. [AKS20a], Cui et al. [CK21b]
- LQ MFG with actor-critic: [FYCW19, uZZMB20], or policy gradient: Wang et al. [WHYW21]
- RL for partially observable MFG: Subramanian et al. [STCP20]
- Mean field RL for multiple types: Subramabian et al. [SPTH20, uZMB22]
- Learning Master policies with deep RL: Perrin et al. [PLP+21a]
- Learning with a single agent: [AFL20, ZKBB23]
- ...

Outline

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Stationary Setting – Reminder

Setting:

- Stationary setting: $N_T = \infty$
- No fixed initial m_0 but a stationary distribution
- Notation: $\mathrm{MF}(\pi) :=$ stationary distribution when using policy π:

$$\mu = P_{\mu,\pi}^{\top} \mu =: \mathcal{P}^{\pi}(\mu)$$

- Player's reward: for player's policy $\pi \in \Delta_A$ and mean field $\mu \in \Delta_S$,

$$J(\pi; \mu) = \mathbb{E}\left[\sum_{n=0}^{\infty} \gamma^n r(x_n, a_n, \mu)\right]$$

where $\gamma \in (0, 1)$ is a discount parameter, and

$$a_n \sim \pi(\cdot|x_n), \qquad x_0 \sim \mu, \quad x_{n+1} \sim p(\cdot|x_n, a_n, \mu), n \geq 0$$

Solution concepts:

- **Stationary MFG Nash equilibrium:** $(\hat{\pi}, \hat{\mu}) \in \Pi \times \Delta_{S \times A}$ s.t.
  1. Best response: $\hat{\pi} \in \mathrm{BR}(\hat{\mu}) := \mathrm{argmax}_{\pi} J(\pi; \hat{\mu})$
  2. Mean field state: $\hat{\mu} = \mathrm{MF}(\hat{\pi})$
- Fixed point: $\hat{\mu} \in \mathrm{MF}(\mathrm{BR}(\hat{\mu}))$

- **Stationary MFC Social optimum:** $\pi^* \in \Pi$ s.t.
  - Optimality: $\pi^* \in \mathrm{argmax}_{\pi^*} J(\pi^*; \mu^{\pi^*})$ where $\mu^{\pi^*} = \mathrm{MF}(\pi^*)$

Evolutive Setting – Reminder

Setting:

- Horizon: $N_T \in \mathbb{N}$ (extensions: p, r depending on n; infinite horizon)
- Fixed initial state distribution: $m_0 \in \Delta_S$
- The MF evolves in time: $\boldsymbol{\mu} = (\boldsymbol{\mu}_n)_{n=0,\dots,N_T} \in \Delta_S^{N_T}$
- Notation $\mathrm{MF}_{m_0,N_T}(\boldsymbol{\pi}) :=$ generated by policy $\boldsymbol{\pi}$ starting from m_0:

$$\boldsymbol{\mu}_0 = m_0, \qquad \boldsymbol{\mu}_{n+1} = P^\top_{\boldsymbol{\mu}_n, \boldsymbol{\pi}_n} \boldsymbol{\mu}_n, \quad n \geq 0$$

- Player's reward: for player's policy $\boldsymbol{\pi} \in \Pi^{N_T}$ and mean field $\boldsymbol{\mu} \in \Delta_S^{N_T}$,

$$J(\boldsymbol{\pi}; \boldsymbol{\mu}) = \mathbb{E}\left[\sum_{n=0}^{N_T} r(x_n, a_n, \boldsymbol{\mu}_n)\right]$$

where $a_n \sim \boldsymbol{\pi}_n(\cdot|x_n), \qquad x_0 \sim m_0, \quad x_{n+1} \sim p(\cdot|x_n, a_n, \boldsymbol{\mu}_n), n \geq 0$

Solution concepts:

- **Evolutive MFG Nash equilibrium:** $(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\mu}}) \in \Pi^{N_T} \times \Delta_S^{N_T}$ s.t.
  1. Best response: $\hat{\boldsymbol{\pi}} \in \mathrm{BR}(\hat{\boldsymbol{\mu}}) := \mathrm{argmax}_{\boldsymbol{\pi}} J(\boldsymbol{\pi}; \hat{\boldsymbol{\mu}})$
  2. Mean field flow: $\hat{\boldsymbol{\mu}} = \mathrm{MF}_{m_0,N_T}(\hat{\boldsymbol{\pi}})$
- Fixed point: $\hat{\boldsymbol{\mu}} \in \mathrm{MF}_{m_0,N_T}(\mathrm{BR}(\hat{\boldsymbol{\mu}}))$

- **Evolutive MFC Social optimum:** $\boldsymbol{\pi}^* \in \Pi^{N_T}$ s.t.
  - Optimality: $\boldsymbol{\pi}^* \in \mathrm{argmax}_{\boldsymbol{\pi}} J(\boldsymbol{\pi}; \boldsymbol{\mu}^{\boldsymbol{\pi}})$ where $\boldsymbol{\mu}^{\boldsymbol{\pi}} = \mathrm{MF}_{m_0,N_T}(\boldsymbol{\pi})$

Evolutive Setting – Infinite Horizon Discounted

Setting:

- Horizon: $N_T = +\infty$; discount $\gamma \in (0, 1)$

Evolutive Setting – Infinite Horizon Discounted

Setting:

- Horizon: $N_T = +\infty$; discount $\gamma \in (0, 1)$
- Notation $\mathrm{MF}_{m_0, N_T}(\boldsymbol{\pi})$:= as before generated by policy $\boldsymbol{\pi}$ starting from m_0 (but now: infinite sequence, $N_T = \infty$)
- Player's reward: for player's policy $\boldsymbol{\pi} \in \Pi^\infty$ and mean field $\boldsymbol{\mu} \in \Delta_S^\infty$,

$$J(\boldsymbol{\pi}; \boldsymbol{\mu}) = \mathbb{E}\left[\sum_{n=0}^{+\infty} \gamma^n r(x_n, a_n, \boldsymbol{\mu}_n)\right]$$

where $a_n \sim \boldsymbol{\pi}_n(\cdot|x_n), \qquad x_0 \sim m_0, \quad x_{n+1} \sim p(\cdot|x_n, a_n, \boldsymbol{\mu}_n), n \geq 0$

Setting:

- Horizon: $N_T = +\infty$; discount $\gamma \in (0, 1)$
- Notation $\mathrm{MF}_{m_0, N_T}(\boldsymbol{\pi}) :=$ as before generated by policy $\boldsymbol{\pi}$ starting from m_0 (but now: infinite sequence, $N_T = \infty$)
- Player's reward: for player's policy $\boldsymbol{\pi} \in \Pi^\infty$ and mean field $\boldsymbol{\mu} \in \Delta_S^\infty$,

$$J(\boldsymbol{\pi}; \boldsymbol{\mu}) = \mathbb{E}\left[\sum_{n=0}^{+\infty} \gamma^n r(x_n, a_n, \boldsymbol{\mu}_n)\right]$$

where $a_n \sim \boldsymbol{\pi}_n(\cdot|x_n), \qquad x_0 \sim m_0, \quad x_{n+1} \sim p(\cdot|x_n, a_n, \boldsymbol{\mu}_n), n \geq 0$

Solution concepts: as before (with infinite sequences, $N_T = \infty$)

From MFC to MFMDP

Let:

$$J^{MFC}(\boldsymbol{\pi}) := J(\boldsymbol{\pi}; \mathrm{MF}_{m_0, N_T}(\boldsymbol{\pi}))$$

MFC problem:

$$\boldsymbol{\pi}^* \in \operatorname*{argmax}_{\boldsymbol{\pi}} J^{MFC}(\boldsymbol{\pi})$$

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Note: in the definition of J using policy $\boldsymbol{\pi}$,

$$\boldsymbol{\mu}_n = \mathcal{L}(x_n)$$

so

$$J^{MFC}(\boldsymbol{\pi}) = \sum_{n=0}^{+\infty} \gamma^n \bar{r}(\bar{a}_n, \boldsymbol{\mu}_n)$$

where $\bar{r}(\bar{a}_n, \boldsymbol{\mu}_n) := \mathbb{E}_{x_n \sim \boldsymbol{\mu}_n, a_n \sim \boldsymbol{\pi}_n(\cdot|x_n)}[r(x_n, a_n, \boldsymbol{\mu}_n)]$

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Intuitively, this is an MDP with state = mean field $\boldsymbol{\mu}_n$: **Mean Field MDP**

Extensions:

- common noise: evolution of $\boldsymbol{\mu}_n$ becomes stochastic
- $\boldsymbol{\pi}$ population-dependent policies: $\boldsymbol{\pi}(\cdot|x_n, \boldsymbol{\mu}_n)$
- **common randomization** [CLT23]: $\boldsymbol{\pi}$ itself can be random, picked according to a central planner's policy $\bar{\boldsymbol{\pi}}$

MFMDP

MFMDP problem:

$$\bar{\boldsymbol{\pi}}^* \in \operatorname*{argmax}_{\bar{\boldsymbol{\pi}}} \bar{J} J(\bar{\boldsymbol{\pi}}; m_0)$$

where

$$\bar{J}(\bar{\boldsymbol{\pi}}; m_0) = \sum_{n=0}^{+\infty} \gamma^n \bar{r}(\bar{a}_n, \boldsymbol{\mu}_n)$$

subject to:

$$\boldsymbol{\mu}_0 = m_0, \qquad \boldsymbol{\mu}_{n+1} = P^\top_{\boldsymbol{\mu}_n, \bar{\boldsymbol{\pi}}_n} \boldsymbol{\mu}_n \text{ (+ noise)}, \quad n \geq 0$$

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- $\bar{V}^*(\mu)$ and $\bar{Q}^*(\mu, \bar{a})$
- Dynamic programming equations [CLT23] (see also [GGWX23] without common noise, and [MP19b] with common noise but no common randomization)
- Need to properly define the class of actions and policies [omitted here; see e.g. [CLT23] for details]

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RL:

- From here, we can re-use existing RL methods for this MDP of mean-field type
- *Question 1: What is the environment?*
- *Question 2: How to deal with the (continuous) state?*

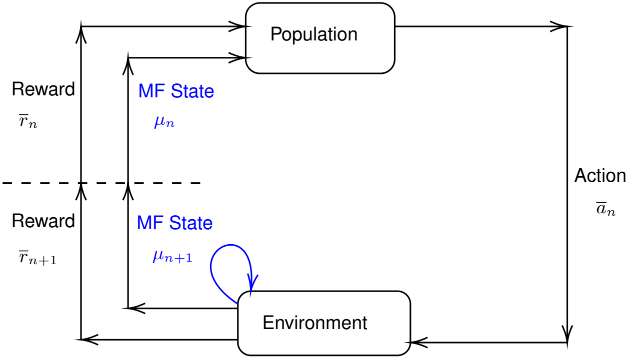
Outline

MFMDP: Environment

Mean field MDP $(\bar{S} = \Delta_S, \bar{A} = \Delta_A^S, \bar{p}, \bar{r}, \gamma)$:

$$\bar{Q}^{\bar{\pi}}(\mu, \bar{a}) = (\bar{\mathcal{B}}^{\mu,\bar{\pi}} \bar{Q}^{\bar{\pi}})(\mu, \bar{a}) = \bar{r}(\mu, \bar{a}) + \gamma \mathbb{E}_{\substack{\mu' \sim \bar{p}(\cdot|\mu, \bar{a}), \\ \bar{a}' \sim \bar{\pi}(\cdot|\mu)}} \left[\bar{Q}^{\bar{\pi}}(\mu', \bar{a}') \right]$$

$\rightarrow$ *What if $\bar{p}$ & $\bar{r}$ are unknown and we can only observe samples $(x', \bar{r}(\mu, \bar{a}))$?*



Mean Field State

How to deal with the MFMDP value functions (and policies)?

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- Option 1: discretize the simplex(es) and then use tabular RL methods
- Option 2: function approximation $Q_\theta(\mu, \bar{a})$ and then use deep RL methods
- Remarks on **policy randomization:**
  - Randomization at the agent level is useful to allow agents to have different trajectories even when start at the same state
  - There exists an optimal policy which is pure at the pop. level [CLT23]
  - But **common randomization** (at the pop. level) helps with exploration

A (Non-exhaustive) Glance at the literature: RL for MFC

RL for Mean Field Control:

- Early works on MDP viewpoint: Gast et al. [GG11, GGLB12]
- Policy optimization for stationary MFC: Subramanian et al. [SM19]
- Policy gradient for LQ MFC [CLT19b, WHYW21] and zero sum mean field type game [CHLT20]
- Multi-time scale for MFC (and MFG): Angiuli et al. [AFL20, AFLZ20, AH21]:
- Mean field MDP: dynamic programming and RL [CLT23, GGWX23, MP19b, GGWX20, CTSK21]
- Decentralized network approach [GGWX21]
- Model based RL for MFC: Pasztor et al. [PBK21]
- . . .

Tabular Q-Learning for MFMDP: Numerical Illustration

Cyber-security example of [KB16]

- MFC viewpoint, MF Q-learning
- pure (population and individual) strategies
- discretization of $\bar{S} = \Delta_S, \bar{A} = \Delta_{S \times A}$

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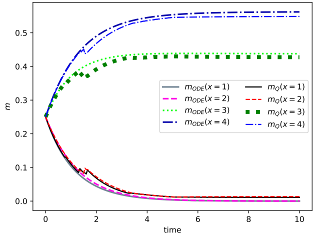
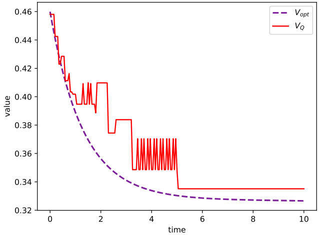
Test 1: $m_0 = (1/4, 1/4, 1/4, 1/4)$



Evolution of m^{m_0} optimally controlled (m_{ODE}) or controlled using the approximate Q-function (m_Q)



V function (V_{opt}) and approximate Q-function (V_Q) along the optimal flow.

(See section 8.1 of [Lau21] and section 6.1 of [CLT23])

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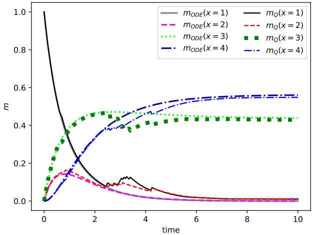
Test 2: $m_0 = (1, 0, 0, 0)$



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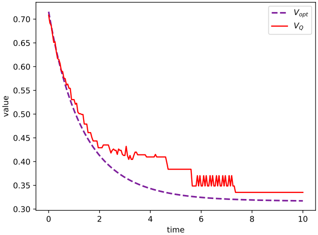

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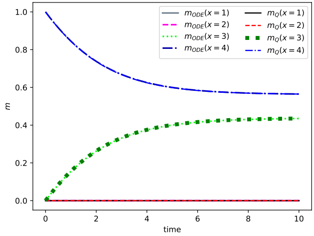
Test 3: $m_0 = (0, 0, 0, 1)$



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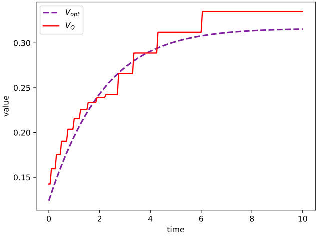

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- But:
  - leads to errors due to projections on the discretized state space
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From Tabular to Deep RL for MFMDP

- Tabular RL is easy to implement and well understood (convergence, etc.)
- But:
  - leads to errors due to projections on the discretized state space
  - not feasible if the number $|S|$ of (individual) states is large, because μ becomes high dimensional
- Instead of discretizing the distribution, we can:
  - replace $\bar{Q}^*$ by a parameterized function, e.g., neural network
  - train it using a deep RL algorithm, e.g., DDPG, ...
- Deep RL for MFMDP: See sections 6.1, 6.2 and 6.3 of [CLT23]

Sample code

Code

Sample code to illustrate: IPython notebook

https://colab.research.google.com/drive/1W8H4EM0bx0RFQFzIaNEcPiEYzG02b0jb?usp=sharing

- Same example as above: MFC for cybersecurity
- Solved using deep RL with population-dependent controls

Another Example: Distribution Planning

- Goal: match a target distribution.
- $S = \{1, \ldots, 10\}$ and $A = \{-1, 0, +1\}$.
- Transitions: $F(x, a, \mu, e, e^0) = x + a + e^0$.
- Cost:
$$f(x, a, \mu) = |a| + \sum_i |\mu(i) - \mu_{\text{target}}(i)|^2.$$
- Here we chose: $\mu_{\text{target}} = (0, 0, 0.05, 0.1, 0.2, 0.3, 0.2, 0.1, 0.05, 0, 0)$.
- No idiosyncratic noise.
- Hence in general it is not possible to match the target distribution unless the agents are allowed to randomize their actions at the individual level.
- We use $(\Delta_A)^S$ for the level-1 action space.
- Without or with common noise $\varepsilon_n^0 \in A$.
- It is not feasible to rely on a tabular method. We show deep RL results.

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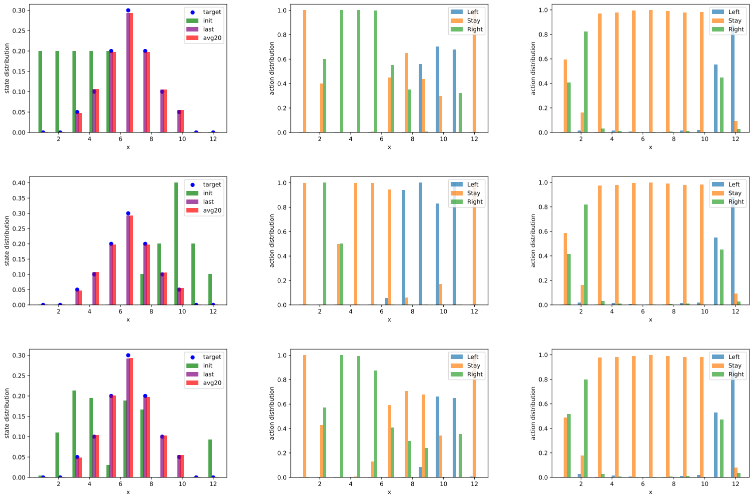
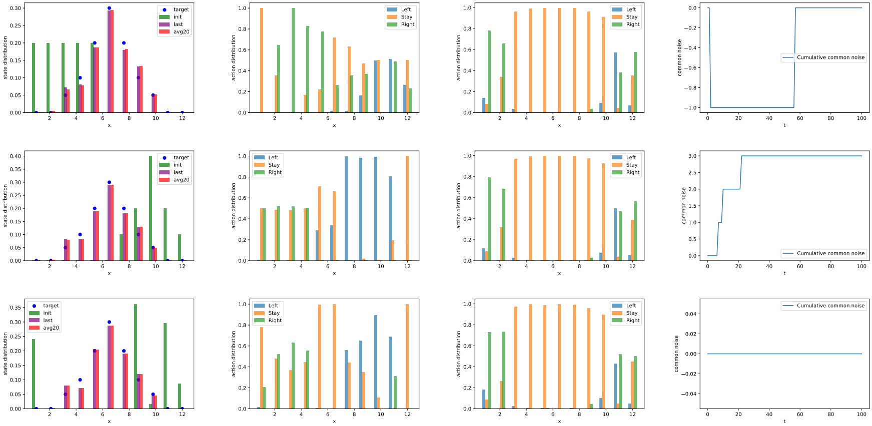


More details in [CLT23]

Another Example: Distribution Planning with Common Noise



More details in [CLT23]

Some Proofs of Convergence

Proof of convergence of RL methods for MFMDP?

- Tabular Q-learning after simplex discretization [CLT23]
- Policy gradient for LQ MFC [CLT19a]
- Still a lot of open questions to study

Outline

Reminder:

- **MFGame:** Fix a distribution μ, compute best response π^{μ}, update μ, ...
- **MFControl:** Fix a policy π, compute induced distribution μ^{π}, update π, ...

Unification of Iterations via Two timescales [AFL20]

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Relaxation: using two-timescale idea

- computing best response $\pi^{\mu} \approx$ many steps of policy improvement
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- rewrite each scheme with 2 nested loops

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Unification: update both π, μ simultaneously but at different rates ρ^{π}, ρ^{μ}

- $\rho^{\pi} < \rho^{\mu} \Rightarrow \pi$ evolves slowly $\Rightarrow$ MFControl
- $\rho^{\pi} > \rho^{\mu} \Rightarrow \mu$ evolves slowly $\Rightarrow$ MFGame

Policy improvement can be implemented through the Q-function for instance:

$$Q(x,a) = f(x,\mu,a) + \sum_{x' \in \mathcal{X}} p(x'|x,\mu,a) \max_{a'} Q(x',a').$$

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The scheme (using ideal updates) can be written as: for $k \geq 0$

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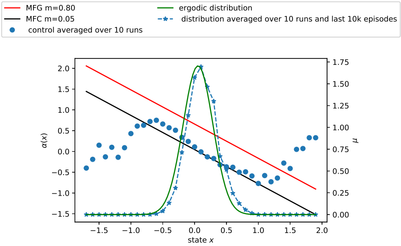
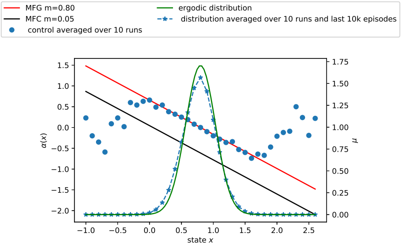
Extension: **sample-based asynchronous** (stochastic approximation [Bor09])

Numerical illustration: Linear-quadratic example

- fixed (quadratic) reward function and (linear) drift function
- the two notions of solutions (MFG/MFC) are different



MFC solution ($\rho^Q < \rho^\mu$)



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- Theory: Proof of convergence [AFLZ23]
- Application: Tuning properly the two learning rates is not trivial!
- Extension: this approach also works for other models, such as **mean field control games (MFCG)** [ADF$^+$22b, ADF$^+$22a]

  $\rightarrow$ MFG where each agent is of mean field type (solves an MFC)

  $\rightarrow$ 3 time scales instead of 2

Outline

Summary

- Settings (static, stationary, evolutive, . . .)
- Solution concepts (Nash, Social opt., . . .)
- Iterative learning methods for MFG (fixed point, fictitious play, . . .)
- Model-free RL methods for MFG (intuition, implementation in OpenSpiel, . . .)
- MFC and Mean Field MDP
- Tabular and Deep RL for MFMDP

Future Directions

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  - ... for for deep RL algorithms
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  - Same questions for tabular RL algorithms (sample complexity, exploration/exploitation, ...)
  - ... for for deep RL algorithms
  - Extension beyond "plain" MFG/MFC
- **Applications:**
  - More efficient implementation of existing methods
  - Contributing to OpenSpiel (more algorithms, more environments, ...)
  - Real-world applications (more realistic model, real data, ...)

Thank you!

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