Numerical Methods for Mean Field Games

Lecture 4
Deep Learning Methods: Part I
MFC and MKV FBSDE

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Outline

1. Introduction

2. Deep Learning for MFC

3. Deep Learning for MKV FBSDE

4. Two Examples of Extensions

5. Conclusion

Summary so far

Numerical methods discussed so far:

- ODE system for LQ setting
- FBPDE system
- FBSDE system

“Classical” Numerical Methods for MFG: Some references

Some methods based on the deterministic approach to MFG/MFC:

- Finite difference & Newton method: [Achdou and Capuzzo-Dolcetta, 2010], [Achdou et al., 2012], . . .
- (Semi-)Lagrangian approach: [Carlini and Silva, 2014, Carlini and Silva, 2015], [Carlini and Silva, 2018], [Calzola et al., 2022], . . .
- Augmented Lagrangian & ADMM: [Benamou and Carlier, 2015], [Andreev, 2017a], [Achdou and Laurière, 2016], . . .
- Primal-dual algo.: [Briceño Arias et al., 2018], [Briceño Arias et al., 2019], . . .
- Gradient descent based methods [Laurière and Pironneau, 2016], [Pfeiffer, 2016], [Lavigne and Pfeiffer, 2022], . . .
- Monotone operators [Almulla et al., 2017], [Gomes and Saúde, 2018], [Gomes and Yang, 2020], . . .
- Policy iteration [Cacace et al., 2021], [Cui and Koeppl, 2021], [Camilli and Tang, 2022], [Tang and Song, 2022], [Laurière et al., 2023], . . .
- Finite elements [Benamou and Carlier, 2015], [Andreev, 2017b], . . .
- Cubature [de Raynal and Trillos, 2015], . . .
- Gaussian processes [Mou et al., 2022], . . .
- Kernel-based representation [Liu et al., 2021], . . .
- Fourier approximation [Nurbekyan et al., 2019], . . .

“Classical” Numerical Methods for MFG: Some references

Some methods based on the probabilistic approach to MFG/MFC:

- Cubature [de Raynal and Trillos, 2015], . . .
- Markov chain approximation: [Bayraktar et al., 2018], . . .
- Probabilistic approach and Picard: [Chassagneux et al., 2019], [Angiuli et al., 2019], . . .
- Probabilistic approach and regression: [Balata et al., 2019], . . .
- . . .

“Classical” Numerical Methods for MFG: Shortcomings

Many of these methods are very efficient and have been analyzed in detail

However, they are usually limited to problems with:

- (relatively) small dimension
- (relatively) simple structure

$\Rightarrow$ motivations to develop machine learning methods (see lectures 4, 5, 6)

Deep learning

- In this lecture and the following one, we will use deep learning to solve MFGs
- At a high level, there are two main ingredients:
  - Approximation using deep neural networks
  - Minimization of a loss function using stochastic gradient descent
- Many variants and refinements, . . .
- See e.g. [LeCun et al., 2015, Goodfellow et al., 2016], . . .

Ingredient 1: Neural Networks

- **Goal:** Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_\xi[\mathbb{L}(\varphi, \xi)]$
- Example: Regression: $\xi = (x, f(x))$ for some f, $\mathbb{L}(\varphi, \xi) = \|\varphi(x) - f(x)\|^2$

- **Goal:** Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_\xi[\mathbb{L}(\varphi, \xi)]$
- Example: Regression: $\xi = (x, f(x))$ for some f, $\mathbb{L}(\varphi, \xi) = \|\varphi(x) - f(x)\|^2$
- **Idea:** Instead of min. over all $\varphi(\cdot)$, min. over parameters θ of $\varphi_\theta(\cdot)$
- Example: Feedforward fully-connected neural network:
  - $\varphi_\theta(\cdot)$
  - with weights & biases $\theta = (\beta^{(k)}, w^{(k)})_{k=1,\dots,\ell}$
  - activation functions $\psi^{(i)}$: sigmoid, tanh, ReLU, . . . ; applied coordinate-wise

$$\underbrace{\varphi_\theta(x)}_{\varphi(\theta, x)} = \psi^{(\ell)}\Big(\beta^{(\ell)} + w^{(\ell)} \dots \psi^{(2)}\Big(\beta^{(2)} + w^{(2)} \underbrace{\psi^{(1)}(\beta^{(1)} + w^{(1)} x)}_{\text{one hidden layer}}\Big) \dots\Big)$$

  - Depth = number of layers; width of a layer = dimension of bias vector

Ingredient 1: Neural Networks – Comments

- Many other architectures (convolutional neural networks, recurrent neural networks, . . .), see e.g. [Leijnen and Veen, 2020]
- Successes of deep learning in many fields: natural language processing, computer vision, drug design, . . . and even games!
- Combination with reinforcement learning (see lecture 6)
- Universal approximation theorems [Cybenko, 1989], [Hornik, 1991], . . .
- Connections with numerical analysis, see e.g. [Després, 2022]

Ingredient 1: Neural Networks – Gradients

Differentiation: can compute partial derivatives by automatic differentiation (AD) at every (θ, x):

- With respect to parameters: $\nabla_\theta \varphi(\theta, x)$

$$\nabla_{\beta^{(\ell)}} \varphi(\theta, x) = \ldots, \qquad \nabla_{w^{(2)}} \varphi(\theta, x) = \ldots$$

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- With respect to state variable: $\nabla_x \varphi(\theta, x)$ can be computed by AD too

$$\partial_{x_1} \varphi(\theta, x) = \ldots$$

  $\Rightarrow$ can be used in PDEs (see lecture 5)

Ingredient 2: Stochastic Gradient Descent

- **Goal:** Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_\xi[\mathbb{L}(\varphi, \xi)]$
- **Parameterization:** $\widetilde{\mathbb{J}}(\theta) := \mathbb{E}_\xi[\widetilde{\mathbb{L}}(\theta, \xi)]$, where $\widetilde{\mathbb{L}}(\theta, \xi) := \mathbb{L}(\varphi_\theta, \xi)$

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- **Setting:** the distribution of ξ is unknown so we cannot compute $\mathbb{E}_\xi$, but
  - we have some samples (i.e. random realizations) of ξ
  - we know $\mathbb{L}$

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Algorithm: Stochastic Gradient Descent

Input: Initial param. θ_0; data $S = (\xi_s)_{s=1,\dots,|S|}$; nb of steps $\mathtt{K}$; learning rates $(\eta^{(\mathtt{k})})_{\mathtt{k}}$

Output: Parameter $\theta^\star$ s.t. $\varphi_{\theta^\star}$ (approximately) minimizes $\widetilde{\mathbb{J}}$

Initialize $\theta^{(0)} = \theta_0$

for $\mathtt{k} = 0, 1, 2, \dots, \mathtt{K} - 1$ **do**

  Pick $s \in S$ randomly

  Compute the gradient $\nabla_\theta \widetilde{\mathbb{L}}(\theta^{(\mathtt{k}-1)}, \xi_s) = \frac{d}{d\theta}\mathbb{L}(\varphi_{\theta^{(\mathtt{k}-1)}}, \xi_s)$

  Set $\theta^{(\mathtt{k})} = \theta^{(\mathtt{k}-1)} - \eta^{(\mathtt{k})}\nabla_\theta \widetilde{\mathbb{L}}(\theta^{(\mathtt{k}-1)}, \xi_s)$

return $\theta^{(\mathtt{K})}$

Ingredient 2: Stochastic Gradient Descent – Comments

- Many variants:
  - Learning rate: `ADAM` (Adaptive Moment Estimation) [Kingma and Ba, 2014], ...
  - Samples: Mini-batches, ...

- Proofs of convergence e.g. using stochastic approximation [Robbins and Monro, 1951], [Borkar, 2009]

- In practice: many details to be discussed, see e.g.[Bottou, 2012]; choice of hyperparameters
  - architecture
  - initialization
  - learning rate
  - loss function
  - ...

Analysis: Error Types

- Consider the task: minimize over φ the population risk:

$$\mathcal{R}(\varphi) = \mathbb{E}_{x,y}[L(\varphi(x), y)]$$

with $x \sim \mu$ and $y = f(x) + \epsilon$ for some noise ϵ where f is unknown

Analysis: Error Types

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- In practice:
  - minimize over a hypothesis class $\mathcal{F}$ of φ
  - finite number of samples, $S = (x_m, y_m)_{m=1,\ldots,M}$: empirical risk:

$$\hat{\mathcal{R}}_S(\varphi) = \frac{1}{M}\sum_{m=1}^{M} L(\varphi(x_m), y_m) \qquad (+\text{ regu})$$

  - finite number of optimization steps, say `k`

Analysis: Error Types

We are interested in:

- Approximation error: Letting $\varphi^* = \operatorname{argmin}_{\varphi \in \mathcal{F}} \operatorname{dist}(\varphi, f)$,

$$\epsilon_{\text{approx}} = \operatorname{dist}(\varphi^*, f)$$

- Estimation error: Letting $\hat{\varphi}_S = \operatorname{argmin}_{\varphi \in \mathcal{F}} \hat{\mathcal{R}}_S(\varphi)$

$$\epsilon_{\text{estim}} = \operatorname{dist}(\hat{\varphi}_S, \varphi^*)$$

- Optimization error: After `k` steps, we get $\varphi_S^{(\mathtt{k})}$;

$$\epsilon_{\text{optim}} = \operatorname{dist}(\varphi_S^{(\mathtt{k})}, \hat{\varphi}_S)$$

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- Optimization error: After $\mathtt{k}$ steps, we get $\varphi_S^{(\mathtt{k})}$;

$$\epsilon_{\text{optim}} = \operatorname{dist}(\varphi_S^{(\mathtt{k})}, \hat{\varphi}_S)$$

- Generalization error of the learnt $\varphi_S^{(\mathtt{k})}$:

$$\epsilon_{\text{gene}} = \epsilon_{\text{approx}} + \epsilon_{\text{estim}} + \epsilon_{\text{optim}}$$

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From optimal control to optimization

- An optimal control is a “temporally extended” optimization problem
- Numerically, we cannot minimize over all possible controls
- We can parameterize the control function
- and then optimize over the parameters
- See e.g. [Gobet and Munos, 2005], [Han and E, 2016], . . .

Stochastic Optimal Control: Approximate Problem

Stochastic optimal control problem:

Minimize over $\alpha(\cdot, \cdot)$

$$J(\alpha(\cdot, \cdot)) = \mathbb{E}\Big[\int_0^T f(X_t, \alpha(t, X_t))\, dt + g(X_T) \Big],$$

with

$$X_0 \sim m_0 \,, \quad dX_t = b(X_t, \alpha(t, X_t))\, dt + \sigma dW_t$$

Stochastic Optimal Control: Approximate Problem

Stochastic optimal control problem: (1) neural network φ_θ,

Minimize over **neural network** parameters θ

$$J(\theta) = \mathbb{E}\Big[\int_0^T f\left(X_t, \varphi_\theta(t, X_t)\right)\, dt + g\left(X_T\right) \Big],$$

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Stochastic Optimal Control: Approximate Problem

Stochastic optimal control problem: (1) neural network φ_θ, (2) time discretization

Minimize over **neural network** parameters θ and N_T time steps

$$J^{N_T}(\theta) = \mathbb{E}\Big[\sum_{n=0}^{N_T-1} f\left(X_n, \varphi_\theta(t_n, X_n)\right) \Delta t + g\left(X_{N_T}\right) \Big],$$

with

$$X_0 \sim m_0\,, \quad X_{n+1} - X_n = b(X_n, \varphi_\theta(t_n, X_n))\Delta t + \sigma \Delta W_n$$

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→ neural network induces an approximation error

→ time discretization induce extra errors

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To implement SGD, at each iteration we pick a sample $\xi = (X_0, \Delta W_0, \dots, \Delta W_{N_T-1})$

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MFC problem:

Minimize over $\alpha(\cdot,\cdot)$

$$J(\alpha(\cdot,\cdot)) = \mathbb{E}\Big[\int_0^T f(X_t, \mu_t, \alpha(t, X_t))\, dt + g(X_T, \mu_T)\Big],$$

where $\mu_t = \mathcal{L}(X_t)$ with

$$X_0 \sim m_0\,, \quad dX_t = b(X_t, \mu_t, \alpha(t, X_t))\, dt + \sigma dW_t$$

MFC problem: (1) Finite pop.,

Minimize over **decentralized** controls $\alpha(\cdot,\cdot)$ with N agents

$$J^N(\alpha(\cdot,\cdot)) = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_0^T f\left(X_t^i, \mu_t^N, \alpha(t, X_t^i)\right)\, dt + g\left(X_T^i, \mu_T^N\right)\Big],$$

where $\mu_t^N = \frac{1}{N}\sum_{j=1}^{N}\delta_{X_t^j}$, with

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MFC problem: (1) Finite pop., (2) neural network φ_θ,

Minimize over **neural network** parameters θ with N agents

$$J^N(\theta) = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_0^T f\left(X_t^i, \mu_t^N, \varphi_\theta(t, X_t^i)\right)\, dt + g\left(X_T^i, \mu_T^N\right)\Big],$$

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MFC problem: (1) Finite pop., (2) neural network φ_θ, (3) time discretization

Minimize over **neural network** parameters θ with N agents and N_T time steps

$$J^{N,N_T}(\theta) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sum_{n=0}^{N_T-1} f\left(X_n^i, \mu_n^N, \varphi_\theta(t_n, X_n^i)\right)\Delta t + g\left(X_{N_T}^i, \mu_{N_T}^N\right)\right],$$

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Note: we aim for a decentralized control, whereas for a general N-agent control problem, the optimal control is not always of this type

- The following kind of convergence result (bound on the **approximation error**) can be proved, see [Carmona and Laurière, 2022]:

Approximation theorem

Under suitable assumptions (in particular regularity of the value function),

$$\left| \inf_{\alpha(\cdot,\cdot)} J(\alpha(\cdot,\cdot)) - \inf_{\theta\in\Theta} J^{N,N_T}(\theta) \right| \leq \epsilon_1(N) + \epsilon_2(\dim(\theta)) + \epsilon_3(N_T)$$

Convergence Analysis

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- The **estimation error** for shallow neural networks can be analyzed using techniques similar to [Carmona and Laurière, 2021]
- The **optimization error** remains to be studied
- Many extensions and refinements to be investigated

Approximation Error Analysis: Main Ingredients of the Proof

Proposition 1 (N agents & decentralized controls):

Under suitable assumptions, there exists a decentralized control α^* s.t. *(d = dimension of X_t)*

$$\left| \inf_{\alpha(\cdot)} J(\alpha(\cdot)) - J^N(\alpha^*(\cdot)) \right| \leq \epsilon_1(N) \in \widetilde{O}\left(N^{-1/d}\right).$$

Proof: propagation of chaos type argument [Carmona and Delarue, 2018]

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Proposition 2 (approximation by neural networks): Under suitable assumptions

There exists a set of parameters $\theta \in \Theta$ for a one-hidden layer $\hat{\varphi}_\theta$ s.t.

$$\left| J^N(\alpha^*(\cdot)) - J^N(\hat{\varphi}_\theta(\cdot)) \right| \leq \epsilon_2(\dim(\theta)) \in O\left(\dim(\theta)^{-\frac{1}{3(d+1)}}\right).$$

Proof: Key difficulty: approximate $v^*(\cdot)$ by $\hat{\varphi}_\theta(\cdot)$ while controlling $\|\nabla \hat{\varphi}_\theta(\cdot)\|$ by $\|\nabla v^*(\cdot)\|$

$\rightarrow$ universal approximation without rate of convergence is not enough

$\rightarrow$ approximation rate for the derivative too, e.g. from [Mhaskar and Micchelli, 1995]

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Proposition 3 (Euler-Maruyama scheme):

For a specific neural network $\hat{\varphi}_\theta(\cdot)$,

$$\left|J^N(\hat{\varphi}_\theta(\cdot)) - J^{N,N_T}(\hat{\varphi}_\theta(\cdot))\right| \leq \epsilon_3(N_T) \in O\left(N_T^{-1/2}\right).$$

Key point: $O(\cdot)$ independent of N and $\dim(\theta)$

Proof: analysis of **strong error rate** for Euler scheme (reminiscent of [Bossy and Talay, 1997])

Implementation

- Key idea: replace optimal control problem by (finite dim.) optimization problem:
  - Loss function = cost: $J^{N,N_T}(\theta) = \mathbb{E}[\mathbb{L}(\varphi_\theta, \xi)]$
  - One sample: $\xi = \left(X_0^j, (\Delta W_n^j)_{n=0,\dots,N_T-1}\right)_{j=1,\dots,N}$

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- Structure:

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  - One sample: $\xi = \left(X_0^j, (\Delta W_n^j)_{n=0,\ldots,N_T-1}\right)_{j=1,\ldots,N}$

$\rightarrow$ can use **Stochastic Gradient Descent**

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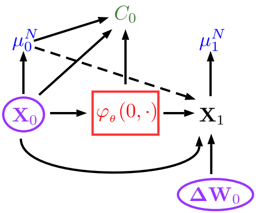

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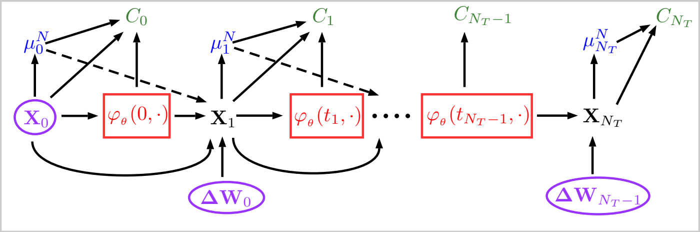

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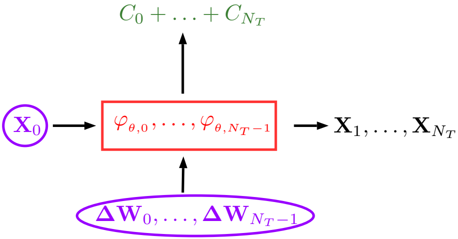


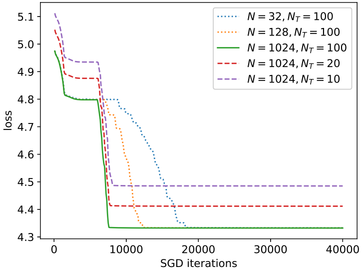
Numerical Illustration 1: LQ MFC

Benchmark to assess **empirical convergence of SGD:** LQ problem with explicit sol.

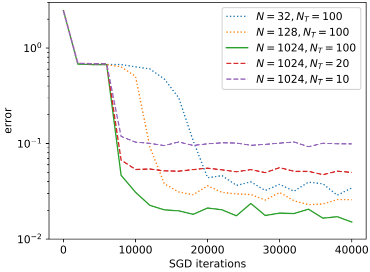
Example: Linear dynamics, quadratic costs of the type

$$f(x,\mu,v) = \underbrace{(\bar{\mu}-x)^2}_{\text{distance to mean position}} + \underbrace{v^2}_{\text{cost of moving}}, \qquad \bar{\mu} = \underbrace{\int \mu(\xi)d\xi}_{\text{mean position}}, \qquad g(x) = x^2$$

Numerical example with $d = 10$ (see [Carmona and Laurière, 2022]):



total cost (= loss function)



L^2-error on the control

Numerical Illustration 2: min-LQ MFC with common noise

The following model is inspired by [Salhab et al., 2015] and [Achdou and Lasry, 2019].

MFC with simple CN:

Dynamics: $dX_t = \phi_t(X_t, \epsilon_t^0)dt + \sigma dW_t$, $\epsilon_t^0 = 0$ until $t = T/2$, and then ξ_1 or ξ_2 w.p. $1/2$

Running cost $|\phi_t(X_t, \epsilon_t^0)|^2$, final cost $(X_T - \epsilon_T^0)^2 + \bar{Q}_T(\bar{m}_T - X_T)^2$

Parameter values: $\sigma = 0.1$, $T = 1$, $\xi_1 = -1.5$, $\xi_2 = +1.5$

Numerical results:

- neural network $\varphi_\theta(t, X_t, \epsilon_t^0)$, taking as an input the common noise
- benchmark solution computed by solving a **system of 6 PDEs** (see [Achdou and Lasry, 2019, Bourany, 2018])

Numerical Illustration 2: min-LQ MFC with common noise

Here the common noise takes one among two values, at time $T/2$.

More details in [Carmona and Laurière, 2022]

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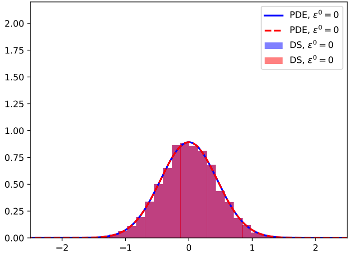

$t = 0$

Until $T/2$: concentrate around mid-point $= 0$

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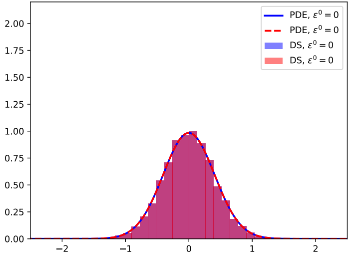

$t = 0.1$

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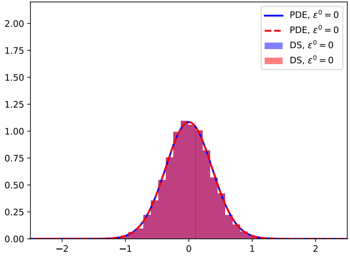

$t = 0.2$

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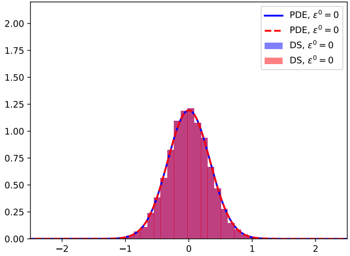

$t = 0.3$

Until $T/2$: concentrate around mid-point $= 0$

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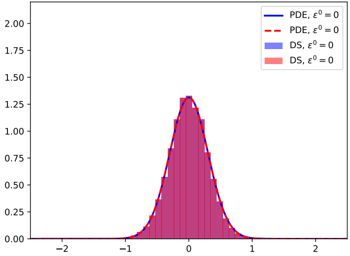

$t = 0.4$

Until $T/2$: concentrate around mid-point $= 0$

More details in [Carmona and Laurière, 2022]

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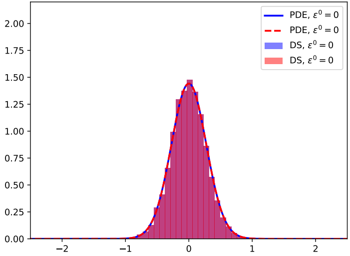

$t = 0.5$

Until $T/2$: concentrate around mid-point $= 0$

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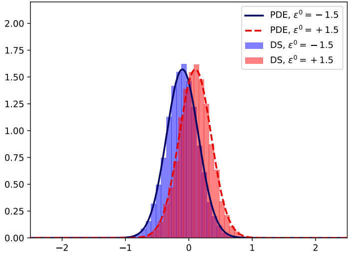

$$t = 0.6$$

Until $T/2$: concentrate around mid-point $= 0$

After $T/2$: move towards the target selected by common noise

More details in [Carmona and Laurière, 2022]

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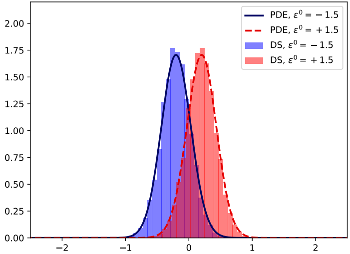

$t = 0.7$

Until $T/2$: concentrate around mid-point $= 0$

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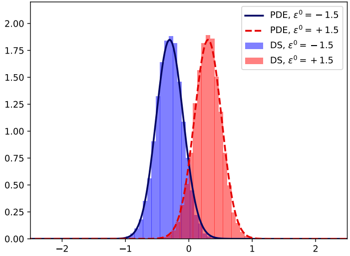

$t = 0.8$

Until $T/2$: concentrate around mid-point $= 0$

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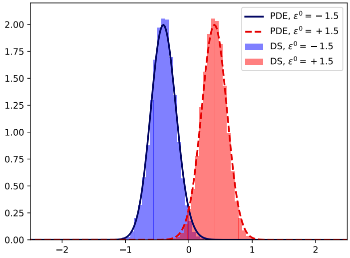

$t = 0.9$

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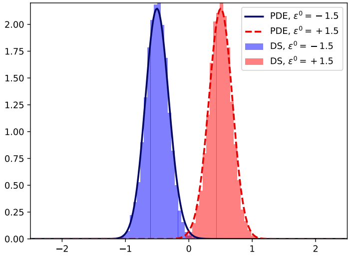

$$t = 1$$

Until $T/2$: concentrate around mid-point $= 0$

After $T/2$: move towards the target selected by common noise

More details in [Carmona and Laurière, 2022]

Numerical Illustration 3: MFC with Interactions Through the Controls

Price Impact Model [Carmona and Lacker, 2015, Carmona and Delarue, 2018]:

- Price process: with ν^{α} = population's distribution over actions,

$$dS_t^{\alpha} = \gamma \int_{\mathbb{R}} a d\nu_t^{\alpha}(a) dt + \sigma_0 dW_t^0$$

- Typical agent's inventory: $dX_t^{\alpha} = \alpha_t dt + \sigma dW_t$
- Typical agent's wealth: $dK_t^{\alpha} = -\big(\alpha_t S_t^{\alpha} + c_{\alpha}(\alpha_t)\big) dt$
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Equivalent problem:

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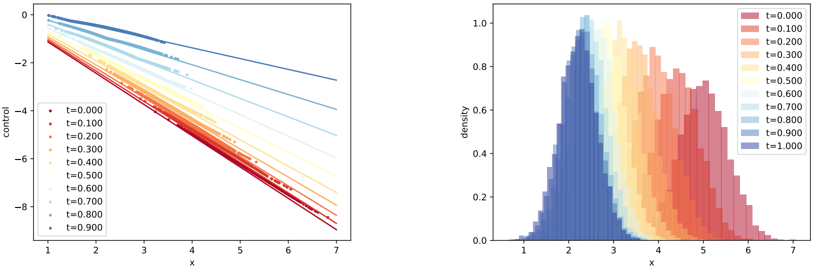
We take: $c_\alpha(v) = \frac{1}{2} c_\alpha v^2$, $c_X(x) = \frac{1}{2} c_X x^2$ and $g(x) = \frac{1}{2} c_g x^2$

Control learnt (left) and associated state distribution (right)



$$T = 1,\ c_X = 2,\ c_\alpha = 1,\ c_g = 0.3,\ \sigma = 0.5,\ \gamma = 0.2$$

See Section 2 in [Carmona and Laurière, 2023] for more details.

Numerical Illustration 3: MFC with Interactions Through the Controls

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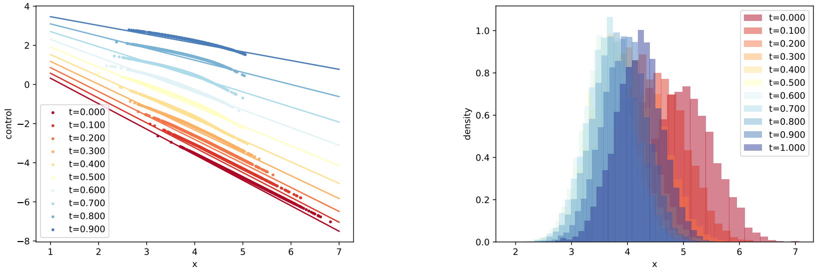

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Sample code

Code

Sample code to illustrate: IPython notebook

`https://colab.research.google.com/drive/1QYWz4Sclw9goRZsbd0uB6wR6a0Uu0a3k?usp=sharing`

- Deep learning for MFC using a direct approach where the control is parameterized as a neural network
- Applied to the price impact model discussed above

Related works

- DL for stochastic control [Gobet and Munos, 2005], [Han and E, 2016], ...
- Various possible implementations; example: 1 NN per time step instead of a single 1 NN as a function of time
- Extensions to finite-player games [Hu, 2021]
- Extension to MFC presented here [Carmona and Laurière, 2022]; see also [Carmona and Laurière, 2023]
- Related works with mean field: [Fouque and Zhang, 2020] (MFC with delay), [Germain et al., 2019], [Agram et al., 2020], [Dayanikli et al., 2023] (with population-dependent controls), ...

Outline

1. Introduction

2. Deep Learning for MFC

3. Deep Learning for MKV FBSDE

4. Two Examples of Extensions

5. Conclusion

Shooting Method for FBSDE

- Goal: solve an FBSDE system
- The backward process has a value Y_0 at time 0, but it is not known
- Try to guess the correct initial condition so that the terminal condition is satisfied
- This yields a new optimal control problem
- See e.g. [Kohlmann and Zhou, 2000], [Sannikov, 2008], ...
- For the new optimal control problem, use deep learning [E et al., 2017]

DeepBSDE (E et al.)

Solutions of sto. control problems can be characterized by **FBSDEs** of the form

$$\begin{cases} dX_t = B(t, X_t, Y_t)dt + dW_t, & X_0 \sim m_0 & \to \text{state} \\ dY_t = -F(t, X_t, Y_t)dt + Z_t \cdot dW_t, & Y_T = G(X_T) & \to \text{control/cost} \end{cases}$$

(stemming from sto. Pontryagin's or Bellman's principle: $F = f$ or $F = \partial_x H$)

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Shooting: Guess Y_0 and $(Z_t)_t$
$\rightarrow$ recover sol. (X, Y, Z) is found by opt. control of 2 **forward** SDEs

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Reformulation as a new **optimal control problem**

Minimize over $y_0(\cdot)$ and $\mathbf{z}(\cdot) = (z_t(\cdot))_{t\geq 0}$

$$\mathfrak{J}(y_0(\cdot), \mathbf{z}(\cdot)) = \mathbb{E}\left[\|Y_T^{y_0,\mathbf{z}} - G(X_T^{y_0,\mathbf{z}})\|^2 \right],$$

under the constraint that $(X^{y_0,\mathbf{z}}, Y^{y_0,\mathbf{z}})$ solve: $\forall t \in [0, T]$

$$\begin{cases} dX_t = B(t, X_t, Y_t)dt + dW_t, & X_0 \sim m_0, \\ dY_t = -F(t, X_t, Y_t)dt + z(t, X_t) \cdot dW_t, & Y_0 = y_0(X_0). \end{cases}$$

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$\to$ New optimal control problem: apply previous method, replacing $y_0(\cdot), z(\cdot, \cdot)$ by NN

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Note: This problem is *not* the original stochastic control problem !

Application to Solve PDEs

This method can be used to solve PDEs [E et al., 2017]

Feynman-Kac formula: correspondence $u(t, X_t) = Y_t$ where

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- X solves the SDE:

$$dX_t = B(t, x)dt + \sigma dW_t$$

- (Y, Z) solves the BSDE:

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- In fact $Z_t = \sigma \partial_x u(t, X_t)$

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This method can be used to solve PDEs [E et al., 2017]

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- Ex. HJB equation. Many variations/extensions

Deep MKV FBSDE

Solutions of MFG (and MFC) can be characterized by **MKV FBSDEs** of the form

$$\begin{cases} dX_t = B(t, X_t, \mathcal{L}(X_t), Y_t)dt + dW_t, & X_0 \sim m_0 & \to \text{state} \\ dY_t = -F(t, X_t, \mathcal{L}(X_t), Y_t)dt + Z_t \cdot dW_t, & Y_T = G(X_T, \mathcal{L}(X_T)) & \to \text{control/cost} \end{cases}$$

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Reformulation as a **MFC problem** [Carmona and Laurière, 2022]

Minimize over $y_0(\cdot)$ and $\mathbf{z}(\cdot) = (z_t(\cdot))_{t\geq 0}$

$$\mathfrak{J}(y_0(\cdot), \mathbf{z}(\cdot)) = \mathbb{E}\Big[\|Y_T^{y_0,\mathbf{z}} - G(X_T^{y_0,\mathbf{z}}, \mathcal{L}(X_T^{y_0,\mathbf{z}}))\|^2 \Big],$$

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$\rightarrow$ New MFC problem: apply previous method, replacing $y_0(\cdot), z(\cdot, \cdot)$ by NN

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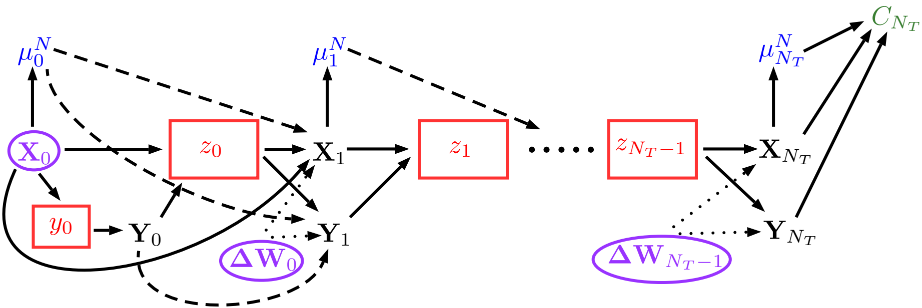
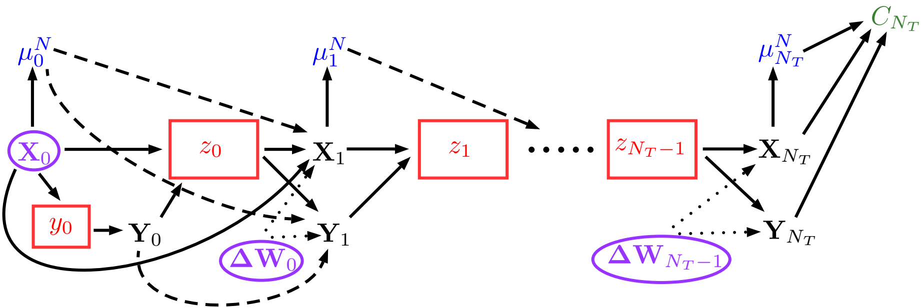
NB: This problem is *not* the original MFG or MFC

Implementation



- **Inputs:** initial positions $\mathbf{X}_0 = (X_0^i)_i$, BM increments: $\mathbf{\Delta W}_n = (\Delta W_n^i)_i$, for all n
- **Loss function:** total cost = C_{N_T} = terminal penalty; state = (X_n, Y_n)
- **SGD** to optimize over the param. θ_y, θ_z of 2 NN for $y_{\theta_y}(\cdot) \approx y_0(\cdot), z_{\theta_z}(\cdot,\cdot) \approx z(\cdot,\cdot)$



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- Alternative implementation: $1 + N_T$ NNs for $y_0(\cdot), z_0(\cdot), \dots, z_{N_T-1}(\cdot)$

Numerical Illustration 1: Comparison with Picard Solver

Example of MKV FBSDE from [Chassagneux et al., 2019] ($\rho =$ coupling parameter)

$$dX_t = -\rho Y_t dt + \sigma dW_t, \qquad X_0 = x_0$$
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Comes from the **MFG** defined by $dX_t^\alpha = \alpha_t dt + dW_t$ and

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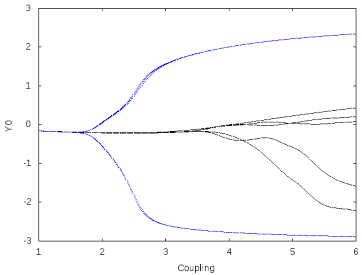
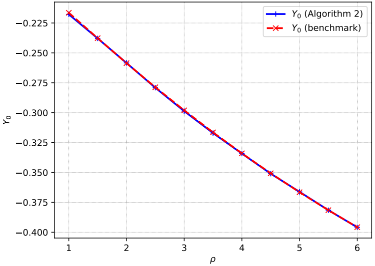


[Chassagneux et al., 2019]



NN (FBSDE system)

More details in [Carmona and Laurière, 2022]

Numerical Illustration 2: LQ MFG with Common Noise

Example: MFG for inter-bank borrowing/lending [Carmona et al., 2015]

X = log-monetary reserve, α = rate of borrowing/lending to central bank, cost:

$$J(\alpha;\bar{m}) = \mathbb{E}\left[\int_0^T \left[\frac{1}{2}\alpha_t^2 - q\alpha_t(\bar{m}_t - X_t) + \frac{\epsilon}{2}(\bar{m}_t - X_t)^2\right] dt + \frac{c}{2}(\bar{m}_T - X_T)^2\right]$$

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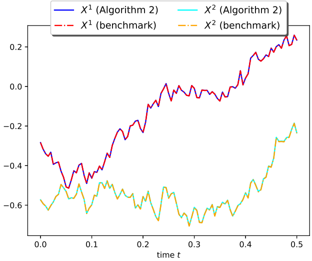
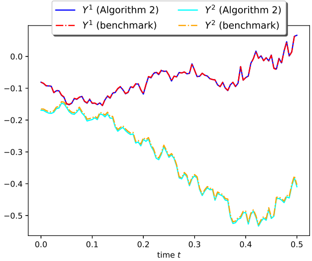
NN for FBSDE system VS (semi) analytical solution (LQ structure)



Samples of X



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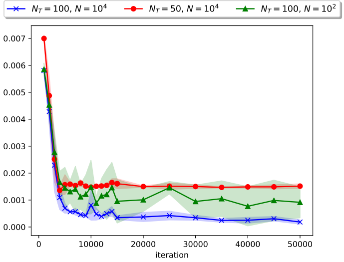
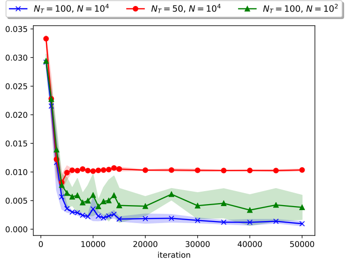


L^2 error on X



L^2 error on Y

More details in [Carmona and Laurière, 2022]

Sample code

Code

Sample code to illustrate: IPython notebook

`https://colab.research.google.com/drive/1w5pMwMxvoVRXFZ1y71-zecyctBTdVl37?usp=sharing`

- Deep learning for MKV FBSDEs
- Applied to the systemic risk model discussed above

Comments

- Convergence of the DeepBSDE method [Han and Long, 2020]
- Extension to finite-player games [Han et al., 2022]
- Analysis of the different types of errors to be done for MKV case
- The new MFC problem is not standard
- Deep learning of MKV FBSDEs as presented here [Carmona and Laurière, 2022]; see also [Carmona and Laurière, 2023]
- Related works on deep learning for MKV FBSDEs: [Fouque and Zhang, 2020] (MFC with delay), [Germain et al., 2019], [Aurell et al., 2022b], ...
- Similar "shooting" strategy can be applied to (infinite-dimensional) ODE systems obtained in graphon games [Aurell et al., 2022a]. Code (Gökçe Dayanıklı):

  `https://github.com/gokce-d/GraphonEpidemics`

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3. Deep Learning for MKV FBSDE

4. Two Examples of Extensions
   - Solving Stackelberg MFG with Deep MKV FBSDE
   - Computing MFC Value Function with DBDP

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Stackelberg MFG

MFG with a Stackelberg (leader-follower) structure:

- A Principal chooses a policy λ
- A population of agents react and form a Nash equilibrium:

$$J^{\boldsymbol{\lambda}}(\boldsymbol{\alpha}, \boldsymbol{\mu}) := \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t, \mu_t; \lambda(t))dt + g(X_T, \mu_T; \lambda(T))\right],$$

- This is an MFG parameterized by λ
- The resulting mean field flow $\hat{\boldsymbol{\mu}}^{\lambda}$ incurs a cost to the principal

$$J^0(\boldsymbol{\lambda}) := \int_0^T f_0(t, \hat{\mu}_t^{\boldsymbol{\lambda}}, \lambda(t))dt + g_0(\hat{\mu}_T^{\boldsymbol{\lambda}}, \lambda(T))$$

Related works: Holmström-Milgrom (1987), Sannikov (2008, 2013), Djehiche-Helgesson (2014), Cvitanić *et al* (2018), Carmona-Wang (2018), Elie *et al* (2019)

DL for Stackelberg MFG

Reminder:

- MFG solution can be characterized using a MKV FBSDE system
- This MKV FBSDE can be rewritten as a control problem
  - 2 forward equations
  - terminal cost

Stackelberg MFG:

- The above terminal cost can be combined with the principal's cost
- We obtain an MFC problem [Elie et al., 2019]
- From here we can apply the methods discussed previously

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For more details, see:

- [Aurell et al., 2022b] with application to epidemics management (finite state MFG): principal gives guidelines (social distancing, etc.) and population reacts
- **Code available** ((Gökçe Dayanıklı)):

  `https://github.com/gokce-d/StackelbergMFG`

- Extension to other Stackelberg MFGs: [Dayanikli and Lauriere, 2023]
- Similarities with DA for mean field optimal transport [Baudelet et al., 2023]

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Social optimum: Mean Field Control

Reminder from lecture 2 about mean field (type) control or control of McKean-Vlasov (MKV) dynamics

Definition (Mean field control (MFC) problem)

α^* is a solution to the MFC problem if it minimizes

$$J^{MFC}(\alpha) = \mathbb{E}\left[\int_0^T f(X_t^\alpha, \alpha_t, m_t^\alpha)dt + g(X_T^\alpha, m_T^\alpha)\right].$$

Main difference with MFG: here not only X but m too is controlled by α.

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Main difference with MFG: here not only X but m too is controlled by α.

Optimality conditions? Several approaches:

- Dynamic programming value function depending on m; value function V
- Calculus of variations taking m as a state; adjoint state u
- Pontryagin's maximum principle for the (MKV process) X; adjoint state Y

Dynamic programming for MFC [Laurière and Pironneau, 2014], [Bensoussan et al., 2015], [Pham and Wei, 2017], [Djete et al., 2022], ...

$\rightarrow$ Algorithm?

DBDP for Non-Mean Field Control

For standard (non-mean field) stochastic optimal control problems, [Huré et al., 2019] have introduced the Deep Backward Dynamic Programming (DBDP):

Idea: learn Y_n and Z_n at each n as functions of X_n, backward in time:

- Initialize $\hat{Y}_{N_T} = g$ and then, for $n = N_T - 1, \dots, 0$, either:
- Version 1: Let $(\hat{Y}_n, \hat{Z}_n)$ = minimizer over (Y_n, Z_n) of:

$$\mathbb{E}\left[|\hat{Y}_{n+1}(X_{n+1}) - Y_n(X_n) - f(t_n, X_n, Y_n(X_n), Z_n(X_n))\Delta t - Z_n(X_n) \cdot \Delta W_{n+1}| \right]$$

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- or Version 2: Let $(\hat{Y}_n, \hat{Z}_n)$ = minimizer over (Y_n, Z_n) of:

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For more details on deep learning methods for (non-mean field) optimal control problems, see e.g. [Germain et al., 2021b]

DBDP for MFC

- Can we apply the same idea to MFC, replacing V by a neural network?
- Main challenge: the value function V takes $m \in \mathcal{P}(\mathbb{R}^d)$ as an input
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- Main challenge: the value function V takes $m \in \mathcal{P}(\mathbb{R}^d)$ as an input
- We need to approximate m
- One possibility:

$$V(t, m_t) \approx \tilde{V}(t, m_t^N) \approx \tilde{V}_\theta(t, X_t^1, \ldots, X_t^N)$$

  where $\tilde{V}_\theta$ is a neural network which is symmetric with respect to the inputs
- See the **lecture 5** for more details
- See [Germain et al., 2021a] for more details about the implementation and [Germain et al., 2022] for the analysis
- See also e.g. [Dayanikli et al., 2023] for different approximations of the population (combined with direct approach instead of DBDP)

Outline

1. Introduction

2. Deep Learning for MFC

3. Deep Learning for MKV FBSDE

4. Two Examples of Extensions

5. Conclusion

Summary

- Two algorithms based on the stochastic approach
- Direct approach without any optimality condition
- DeepBSDE: recasting (MKV) FBSDEs as control problems
- Many possible extensions and variations
- Many open questions for mathematicians (proofs of approximation, rates of convergence, . . .)
- Some surveys on DL for control/games:
  [Germain et al., 2021b, Carmona and Laurière, 2023, Hu and Laurière, 2023]

Next lecture: deep learning methods for the PDE approach

Thank you for your attention

Questions?

Feel free to reach out: mathieu.lauriere@nyu.edu

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