Numerical Methods for Mean Field Games

Lecture 5
Deep Learning Methods: Part II
FBPDEs and Master equations

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Outline

Recap of Lecture 4

- Background on deep learning (DL)
- DL for MFC using a direct approach
- DL for MKV FBSDEs using a “shooting method”
- Extensions
- What about DL for the PDE approach to MFG/MFC?

Outline

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- Look for $\varphi : \mathbb{R} \ni x \mapsto \varphi(x) \in \mathbb{R}$ s.t.

$$\begin{cases} F(x, \varphi(x), \varphi'(x), \dots) = 0, & x \in [a, b] \\ G(a, \varphi(a), \varphi'(a), \dots) = 0 \end{cases}$$

Solving ODEs with Neural Networks

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- Rephrase as minimization problem: minimizer over θ

$$L(\theta) = \mathbb{E}_{X \sim \mathcal{U}([a,b])} \left[|F(X, \varphi_\theta(X), \varphi_\theta'(X), \dots)|^2 \right] + |G(a, \varphi_\theta(a), \varphi_\theta'(a), \dots)|^2$$

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- Use SGD
- Note: we solve and ODE without discretizing time!

Numerical Illustration

Application to the following ODE:

$$\begin{cases} F(x, \varphi(x), \varphi'(x)) = \varphi'(x) - (x - \varphi(x)), & x \in [0, 5] \\ \varphi(0) = 1 \end{cases}$$

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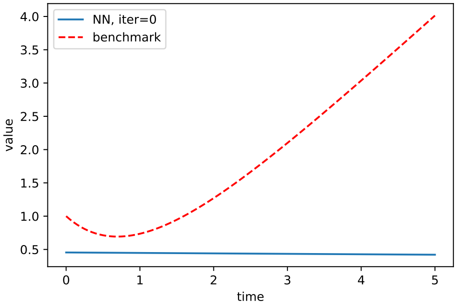

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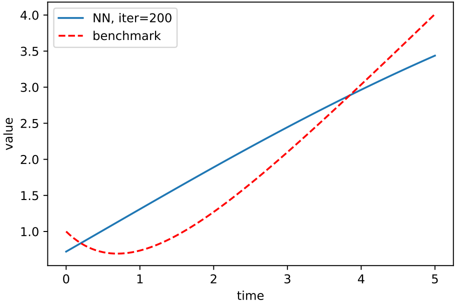

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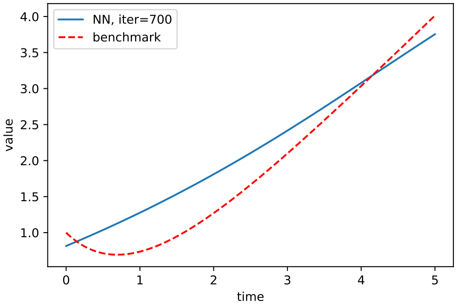

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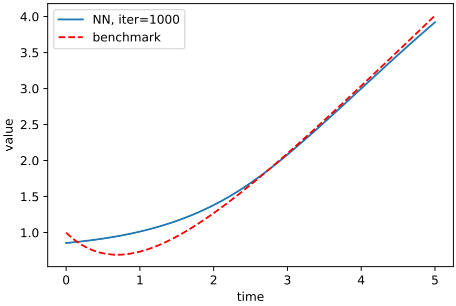

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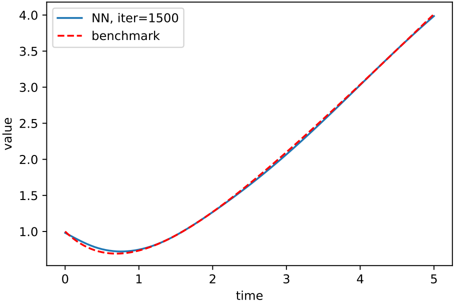

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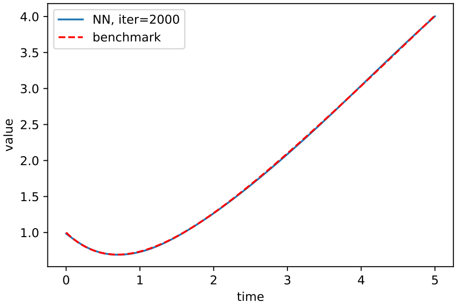

Sample code

Code

Sample code to illustrate: IPython notebook

`https://colab.research.google.com/drive/1pHAKlcRwpeMwzTFI7CcEi3NI5uo0cVqE?usp=sharing`

- ODE
- Solved by DGM

Outline

Deep Galerkin Method (DGM), proposed by [Sirignano and Spiliopoulos, 2018]

- Look for $\varphi : \mathbb{R}^d \ni x \mapsto \varphi(x) \in \mathbb{R}$ s.t.

$$\begin{cases} F(x, \varphi(x), D\varphi(x), D^2\varphi(x), \dots) = 0, & x \in \Omega \\ G(x, \varphi(x), D\varphi(x), D^2\varphi(x), \dots) = 0, & x \in \partial\Omega \end{cases}$$

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- Rephrase as minimization problem: minimizer over θ

$$\begin{aligned} L(\theta) = {} & \mathbb{E}_{X \sim \mathcal{U}(\Omega)} \left[\left| F(X, \varphi_\theta(X), D\varphi_\theta(X), D^2\varphi_\theta(X), \dots) \right|^2 \right] \\ & + \mathbb{E}_{Y \sim \mathcal{U}(\partial\Omega)} \left[\left| G(Y, \varphi_\theta(Y), D\varphi_\theta(Y), D^2\varphi_\theta(Y), \dots) \right|^2 \right] \end{aligned}$$

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DGM: Comments

Remarks on the implementation:

- Choice of distribution:
  - influences training and generalization
  - may depend on the problem (e.g., some regions are more important than others)
- Boundary conditions:
  - need to balance their importance with the PDE residual; can be challenging
  - can sometimes imposed by construction
- Higher order derivatives computation:
  - in principle, can be computed automatically but costly in high dimension
  - approximations are possible, see [Sirignano and Spiliopoulos, 2018] for an approximation of second order derivatives
- Choice of architecture:
  - in low dimension, feedforward fully connected networks work
  - in high dimension, they seem inefficient; [Sirignano and Spiliopoulos, 2018] proposed a specific architecture
- Other DL methods for PDEs e.g. [Raissi et al., 2019]

DGM Architecture

- Let $\vec{x} = (t, x)$ be the input
- Architecture: $L + 1$ hidden layers ($\odot$ denotes element-wise multiplication):

$$
\begin{aligned}
S^1 &= \sigma(W^1 \vec{x} + b^1), \\
Z^\ell &= \sigma(U^{z,\ell} \vec{x} + W^{z,\ell} S^\ell + b^{z,\ell}), \quad \ell = 1, \ldots, L, \\
G^\ell &= \sigma(U^{g,\ell} \vec{x} + W^{g,\ell} S^1 + b^{g,\ell}), \quad \ell = 1, \ldots, L, \\
R^\ell &= \sigma(U^{r,\ell} \vec{x} + W^{r,\ell} S^\ell + b^{r,\ell}), \quad \ell = 1, \ldots, L, \\
H^\ell &= \sigma(U^{h,\ell} \vec{x} + W^{h,\ell} (S^\ell \odot R^\ell) + b^{h,\ell}), \quad \ell = 1, \ldots, L, \\
S^{\ell+1} &= (1 - G^\ell) \odot H^\ell + Z^\ell \odot S^\ell, \quad \ell = 1, \ldots, L, \\
f(t, x; \theta) &= W S^{L+1} + b,
\end{aligned}
$$

- The parameters are

$$
\theta = \left\{ W^1, b^1, \left(U^{\alpha,\ell}, W^{\alpha,\ell}, b^{\alpha,\ell} \right)_{\ell=1,\ldots,L, \alpha \in \{z,g,r,h\}}, W, b \right\}.
$$

- The number of units in each layer is M and $\sigma : \mathbb{R}^M \to \mathbb{R}^M$ is an element-wise nonlinearity:

$$
\sigma(z) = \Big(\phi(z_1), \phi(z_2), \ldots, \phi(z_M) \Big),
$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a nonlinear activation function.

MFG PDE system

Reminder: (m,u) solving, on $[0,T]\times\mathbb{T}^d$,

$$\begin{cases} 0 = -\dfrac{\partial u}{\partial t}(t,x) - \nu\Delta u(t,x) + H(x, m(t,\cdot), \nabla\, u(t,x)) \\ 0 = \dfrac{\partial m}{\partial t}(t,x) - \nu\Delta m(t,x) - \mathrm{div}\left(m(t,\cdot)\partial_p H(\cdot, m(t), \nabla\, u(t,\cdot))\right)(x) \\ \ u(T,x) = g(x, m(T,\cdot)), \qquad m(0,x) = m_0(x) \end{cases}$$

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\end{cases}
$$

Or ergodic version: (m, u, λ) on $\mathbb{T}^d$

$$
\begin{cases}
0 = -\nu \Delta u(x) + H(x, m(\cdot), \nabla u(x)) + \lambda \\
0 = -\nu \Delta m(x) - \operatorname{div}\left(m(\cdot)\partial_p H(\cdot, m, \nabla u(\cdot))\right)(x) \\
\quad \displaystyle\int u(x)dx = 0, \qquad \int m(x)dx = 1, m > 0
\end{cases}
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See [Lasry and Lions, 2007], Chapter 7 in [Bensoussan et al., 2013]

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See [Lasry and Lions, 2007], Chapter 7 in [Bensoussan et al., 2013]

There are analogous PDE systems for MFC problems

Inspired by [Almulla et al., 2017]

Example

Ergodic MFC with explicit solution on $\mathbb{T}^d$. Take:

$$f(x, m, \alpha) = \frac{1}{2}|\alpha|^2 + \tilde{f}(x) + \ln(m(x)),$$

with

$$\tilde{f}(x) = 2\pi^2 \left[-\sum_{i=1}^{d} c \sin(2\pi x_i) + \sum_{i=1}^{d} |c \cos(2\pi x_i)|^2 \right] - 2 \sum_{i=1}^{d} c \sin(2\pi x_i),$$

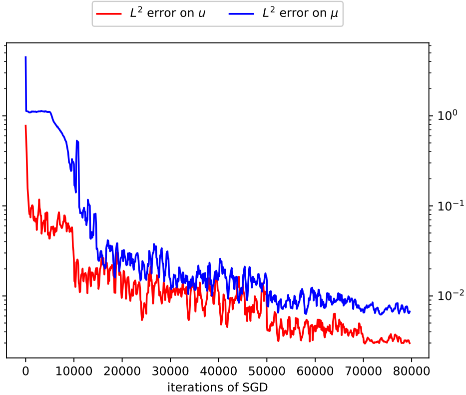
then the solution is given by

$$u(x) = c \sum_{i=1}^{d} \sin(2\pi x_i), \qquad m(x) = \frac{e^{2u(x)}}{\int e^{2u}}$$

Numerical Illustration 1: Ergodic Example with Explicit Solution

Numerical experiments in dimension $d = 10$

Error vs SGD iterations:



Relative L^2 error on u and m

More details in [Carmona and Laurière, 2021a]

Example of MFG without explicit solution on $\mathbb{T}^d$ inspired by [Achdou and Capuzzo-Dolcetta, 2010]

Example

Take:

$$f(x, m, \alpha) = \frac{1}{2}|\alpha|^2 + \tilde{f}(x) + |m(x)|^2,$$

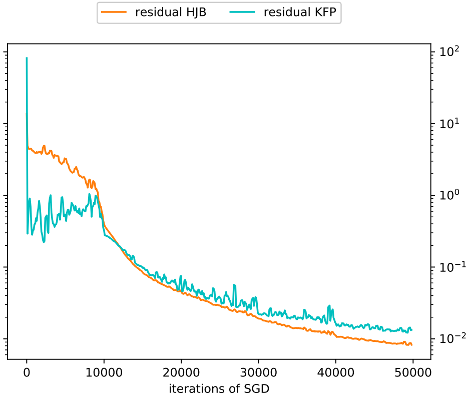
with

$$\tilde{f}(x) = 2\pi^2 c \sum_{i=1}^{d} [\sin(2\pi x_i) + \cos(2\pi x_i)]$$

Numerical Illustration 2: Ergodic Example without Explicit Solution

Numerical experiments in dimension $d = 30$

PDE residuals (training loss) vs SGD iterations:



L^2 norm of residuals for HJB and KFP

More details in [Carmona and Laurière, 2021a]

Sample code

Code

Sample code to illustrate: IPython notebook

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- Ergodic mean field PDE system
- Solved by DGM

Numerical Illustration 3: Crowd Trading

Example

Model of crowd trading by [Cardaliaguet and Lehalle, 2018]:

$$\begin{cases} dS_t^{\bar{\nu}} = \gamma \bar{\nu}_t dt + \sigma dW_t & \text{(price)} \\ dQ_t^{\alpha} = \alpha_t dt & \text{(player's inventory)} \\ dX_t^{\alpha,\bar{\nu}} = -\alpha_t (S_t^{\bar{\nu}} + \kappa \alpha_t) dt & \text{(player's wealth)} \end{cases}$$

Objective: given $(\bar{\nu}_t)_t$, maximize

$$\mathbb{E}\left[X_T^{\alpha,\bar{\nu}} + Q_T^{\alpha} S_T^{\bar{\nu}} - A|Q_T^{\alpha}|^2 - \phi \int_0^T |Q_t^{\alpha}|^2 dt \right]$$

where: $\phi, A > 0 \Rightarrow$ penalty for holding inventory

Numerical Illustration 3: Crowd Trading

Ansatz (see [Cartea and Jaimungal, 2016]):

$$V(t,x,s,q) = x + qsu(t,q), \quad \hat{\alpha}_t(q) = \frac{\partial_q u(t,q)}{2\kappa}$$

where $u(\cdot)$ solves

$$-\gamma\bar{\nu}q = \partial_t u - \phi q^2 + \sup_{\alpha}\{\alpha\partial_q u - \kappa\alpha^2\}, \qquad u(T,q) = -Aq^2$$

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Mean field term: at equilibrium

$$\bar{\nu}_t = \int \hat{\alpha}_t(q) \hat{m}(t,dq) = \int \frac{\partial_q \hat{u}(t,q)}{2\kappa} \hat{m}(t,dq),$$

where $\hat{m}$ solves the KFP equation:

$$m(0,\cdot) = m_0, \qquad \partial_t m + \partial_q \left(m \frac{\partial_q \hat{u}(t,q)}{2\kappa} \right) = 0$$

Numerical Illustration 3: Crowd Trading

Reduced forward-backward PDE system:

$$\begin{cases} 0 = -\partial_t u(t,q) + \phi q^2 - \dfrac{|\partial_q u(t,q)|^2}{4\kappa} = \gamma \bar{\nu}_t q \\ 0 = \partial_t m(t,q) + \partial_q \left(m(t,q) \dfrac{\partial_q u(t,q)}{2\kappa} \right) \\ \bar{\nu}_t = \displaystyle\int \frac{\partial_q u(t,q)}{2\kappa} m(t,q) dq \\ m(0,\cdot) = m_0, u(T,q) = -Aq^2. \end{cases}$$

Note: the interactions are through the action distribution
$\Rightarrow$ yields a non-local term involving both u and m

It can be estimated e.g. by Monte Carlo samples (for a fixed t, sample various q)

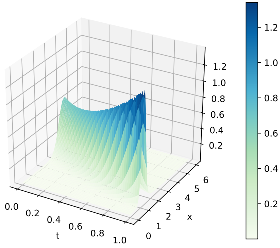
[Al-Aradi et al., 2019] applied DGM to this model.

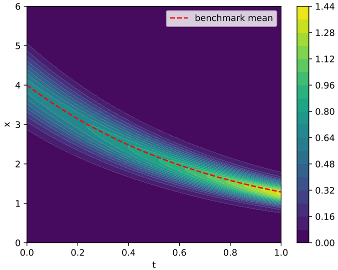
The results presented below are from [Carmona and Laurière, 2021b]

Numerical Illustration 3: Crowd Trading

Numerical results by DGM versus ODE solution

Evolution of m:





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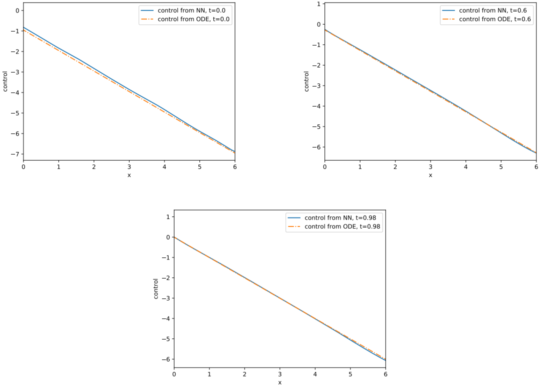
Numerical Illustration 3: Crowd Trading

Numerical results by DGM versus ODE solution

Evolution of equilibrium control $\hat{\alpha}$:



More details in [Carmona and Laurière, 2021b]

Numerical Illustration 3: Crowd Trading

- Convergence of DGM discussed in [Sirignano and Spiliopoulos, 2018]
  - By density, there exists a sequence of NN which approximates the solution and minimizes the DGM loss
  - Conversely, if the DGM loss is small, then the NN is close to the solution
- Similar analysis is possible for MFGs, see e.g. [Luo and Zheng, 2022]
- Variations and improvements, see e.g. [Reisinger et al., 2021]
- Obtaining (good) rates of convergence is challenging, even just for the approximation error
- Understanding the full generalization error remains challenging
- Application to other settings, e.g. mean field optimal transport [Baudelet et al., 2023], and the finite-state master equation (next section)

Outline

Master Equation

- Reminder: equilibrium: $(u, \mu) =$ sol. starting with m_0 at $t = 0$
- Idea: express the **value function** of a typical player as $u(t, x) = \mathcal{U}(t, x, \mu_t)$

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- How can we compute $\mathcal{U}$?

Outline

Finite state MFG:

- Finite state space $\mathcal{X}$
- $\mu \in \Delta^{|\mathcal{X}|}$
- $\dot{\mu}_t = \mu_t Q(\mu_t)$, $Q =$ transition rate matrix

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Master PDE for $\mathcal{U}$:

$$\begin{cases} \mathcal{U}(T, x, \mu) = g(x, \mu) \\ -\partial_t \mathcal{U}(t, x, \mu) = \underbrace{H^*(t, x, \mu, \mathcal{U}(t, \cdot, \mu))}_{\text{Hamiltonian}} + \sum_{x' \in \mathcal{X}} \underbrace{\bar{Q}^*(t, \mu, \mathcal{U}(t, \cdot, \mu))(x')}_{\text{avg transition}} \underbrace{\frac{\partial \mathcal{U}(t, \cdot, \mu)}{\partial \mu(x')}}_{\text{classical deriv.}} \end{cases}$$

for $(t, x, \mu) \in [0, T] \times \mathcal{X} \times \Delta^{|\mathcal{X}|}$

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Numerical solution using the DGM described above

Example 1: Cyber-Security Model

Example (Cyber-security model [Kolokoltsov and Bensoussan, 2016])

- **State space:** $\mathcal{X} = \{DI, DS, UI, US\}$
  - defended/unprotected
  - infected/susceptible
- **Actions:** want to switch level of protection; event happens at rate $\alpha\lambda$
  - $\alpha = 1$ (want to switch level of protection)
  - or 0 (happy)
- **Time:** continuous time, finite time horizon T

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$$\dot{\mu}(t) = \mu(t) \underbrace{\begin{pmatrix} \dots & q_{\text{rec}}^{D} & \alpha\lambda & 0 \\ q_{\text{inf}}^{D} + \beta_{\text{D}}(\mu_{DI}(t) + \mu_{UI}(t)) & \dots & 0 & \alpha\lambda \\ \alpha\lambda & 0 & \dots & q_{\text{rec}}^{U} \\ 0 & \alpha\lambda & q_{\text{inf}}^{U} + \beta_{\text{U}}(\mu_{UI}(t) + \mu_{DI}(t)) & \dots \end{pmatrix}}_{\text{transition rates}}$$

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- **Running cost:**

$$k_D 1_{\{DI,DS\}} + k_I 1_{\{DI,UI\}} = \text{cost of defense} + \text{penalty for being infected}$$

- **Terminal cost:** 0

Example 1: Cyber-Security Model

We apply the DGM. See [Laurière, 2021] for more details.

- Neural network: $\mathcal{U}_\theta$ to approximate $\mathcal{U}$
- Samples: Pick points $(t, x, \mu) \in [0, T] \times \mathcal{X} \times \Delta^{|\mathcal{X}|}$
- Loss: PDE residual + terminal condition

Comparison:

- $\mathcal{U}_\theta(t, x, \mu(t, \cdot))$
- $\mu(t, x)$, $u(t, x)$: finite state space $\rightarrow$ forward-backward ODE system

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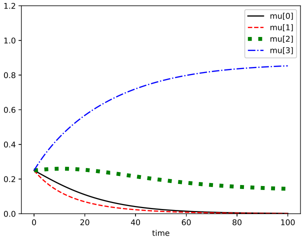
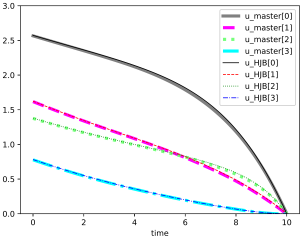
Test 1: $m_0 = (1/4, 1/4, 1/4, 1/4)$



Evolution of μ



Evolution of $u, \mathcal{U}$

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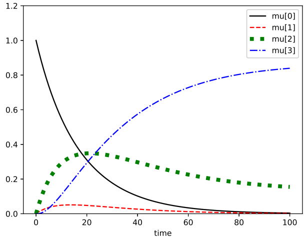
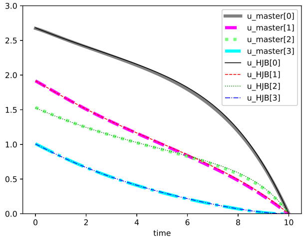
Test 2: $m_0 = (1, 0, 0, 0)$



Evolution of μ



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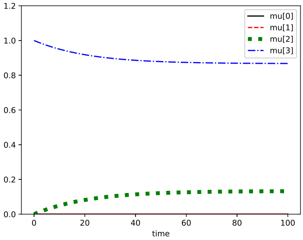
Test 3: $m_0 = (0, 0, 0, 1)$



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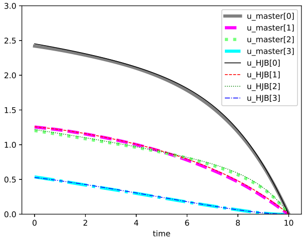
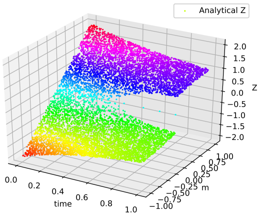


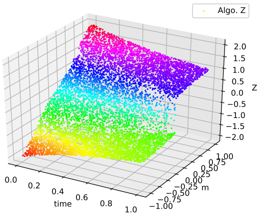
Evolution of $u, \mathcal{U}$

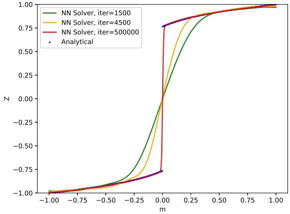
Example 2: Entropic solution

Example of a 2-state MFG [Cecchin et al., 2019] with

- multiple solutions to the master equation
- a unique one is an entropic solution







More details in [Laurière, 2021], section 7.2

Extensions

Some ongoing works:

- Analysis of the DGM convergence for finite-state master equation (ongoing work with Asaf Cohen and Ethan Zell)

- Application to (continuous space) macroeconomic models, joint work with Jonathan Payne and Sebastian Merkel. Working draft on Jonathan's webpage.

Outline

Master Bellman Equation for MFC

- **MFC problem** in **continuous space** with common noise:

$$J^{MFC}(\alpha) = \mathbb{E}\Big[\int_0^T f(X_t, \mathbb{P}^0_{X_t}, \alpha_t)dt + g(X_T, \mathbb{P}^0_{X_T})\Big].$$

subj. to: $dX_t = b(X_t, \mathbb{P}^0_{X_t}, \alpha_t)dt + \sigma dW_t + \sigma_0 dW^0_t$,
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- **Master Bellman equation** in the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$:

$$\begin{cases} \partial_t V + \mathcal{F}(\mu, V, \partial_\mu V, \partial_x \partial_\mu V, \partial^2_\mu V) & = & 0, & (t,\mu) \in [0,T) \in \mathcal{P}_2(\mathbb{R}^d) \\ V(T,\mu) & = & \mathcal{G}(\mu), & \mu \in \mathcal{P}_2(\mathbb{R}^d), \end{cases}$$

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where:

- $\partial_\mu V(\mu)(.) : \mathbb{R}^d \to \mathbb{R}^d$, $\partial_x\partial_\mu V(\mu)(.) : \mathbb{R}^d \to \mathbb{S}^d$, $\partial^2_\mu V(\mu)(.,.) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{S}^d$, are the L-derivatives of V on $\mathcal{P}_2(\mathbb{R}^d)$ (see [Carmona and Delarue, 2018], Chapter 5)
- and

$$\mathcal{F}(\mu, y, Z(.), \Gamma(.), \Gamma_0(.,.)) = \int_{\mathbb{R}^d} h(x,\mu,Z(x),\Gamma(x))\mu(dx) + \int_{\mathbb{R}^d\times\mathbb{R}^d} \tfrac{1}{2}\mathrm{tr}\Big(\sigma_0\sigma_0^\intercal \Gamma_0(x,x')\Big)\mu(dx)\mu(dx'),$$

$$\mathcal{G}(\mu) = \int_{\mathbb{R}^d} g(x,\mu)\mu(dx),$$

$$h(x,\mu,z,\gamma) = \inf_{a\in A}\Big[b(x,\mu,a).z + \frac{1}{2}\mathrm{tr}\Big(\sigma\sigma^\intercal\gamma\Big) + f(x,\mu,a)\Big].$$

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- How can we solve the Bellman PDE and compute V?
- Idea: approximate V by a NN and use backward induction
- Challenge: How can we represent μ and input it to the neural network?

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$$v^N(t, x, x^1, \dots, x^N) = V^N(t, x, \mu^N) \xrightarrow[N \to +\infty]{} V(t, x, \mu^N)$$

- Approximate $V(t, x, \cdot)$ by a **symmetric** function of N inputs (N large)

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- Approximate $V(t,x,\cdot)$ by a **symmetric** function of N inputs (N large)
- **Symmetric Neural Networks:**
  - Symmetry by construction; e.g. with a sum:

$$(x^i)_{i=1,\dots,N} \mapsto \sum_{i=1}^N \psi_\omega(x^i) \mapsto \varphi_\theta\left(\sum_{i=1}^N \psi_\omega(x^i)\right)$$

  - DeepSets [Zaheer et al., 2017], PointNet [Qi et al., 2017], ...

Deep Backward Dynamic Programming for MFC

Deep Learning for MFC with DPP and Symmetric NN [Germain et al., 2021a]

- **Symmetric NN:** $\mathcal{V}(t, x^1, \dots, x^N)$
- **D-Symmetric NN:** sym. except in one space variable:

$$\mathcal{Z}(x^1, \dots, x^N, x^i) \leftrightarrow \partial_{x^i} \mathcal{V}(x^1, \dots, x^N) = \frac{1}{N} \partial_\mu \mathcal{V} \left(\frac{1}{N} \sum_j x^j \right) (x^i)$$

Output: $(\widehat{\mathcal{V}}_n, \widehat{\mathcal{Z}}_n)_{n=0,\dots,N_T}$ s.t. $\widehat{\mathcal{V}}_n(\underline{x}) \approx V(t_n, \mu^N_{\underline{x}})$,
$\widehat{\mathcal{Z}}_n(\underline{x}, x^i) \approx \frac{1}{N} \partial_\mu V(t_n, \mu^N_{\underline{x}})(x^i)$

1. Set $\widehat{\mathcal{V}}_{N_T}(\cdot) = G(\cdot)$
2. **for** $n = N_T - 1, N_T - 2, \dots, 1, 0$ **do**
3. Compute $(\widehat{\mathcal{V}}_n, \widehat{\mathcal{Z}}_n)$ as a minimizer of:

$$(\mathcal{V}_n, \mathcal{Z}_n) \mapsto \mathbb{E}\Big| \widehat{\mathcal{V}}_{n+1}(\mathbf{X}_{n+1}) - \mathcal{V}_n(\mathbf{X}_n) + H\big(t_n, \mathbf{X}_n, \mathcal{V}_n(\mathbf{X}_n), \mathbf{Z}_n(\mathbf{X}_n)\big)\Delta t - \sum_{i=1}^N \sum_{j=0}^N \big(\mathcal{Z}_n(\mathbf{X}_n, X_n^i)\big)^\intercal \sigma_{ij} \Delta W_n^j \Big|^2,$$

where $\widehat{\mathcal{V}}_n$ is a sym. NN, $\widehat{\mathcal{Z}}_n$ is a D-sym. NN, $H =$ sym. version of h

4. **return** $(\widehat{\mathcal{V}}_n, \widehat{\mathcal{Z}}_n)_{n=0,\dots,N_T}$

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return $(\widehat{\mathcal{V}}_n, \widehat{\mathcal{Z}}_n)_{n=0,\dots,N_T}$

See [Germain et al., 2021a] for numerical results and more details about the implementation, and [Germain et al., 2022] for the analysis

Outline

1. Introduction

2. Deep Galerkin Method for MFG PDEs

3. Master Equation

4. Conclusion

Summary

- Deep Galerkin Method principle
  - Application to solve FB PDE system
  - Application to solve finite-state Master equations
- Deep Backward Dynamic Programming & symmetric NN
  - Application to compute the value function of MFC
- Many open questions for mathematicians (proofs of approximation, rates of convergence, . . .)

More references

The presentation in this lecture and the previous one is not exhaustive. Other works, such as: [Ruthotto et al., 2020], and works on the connection between (variational) MFGs and Generative Adversarial Networks (GANs) [Cao et al., 2020], [Lin et al., 2020].

Surveys on deep learning for:

- PDEs [Beck et al., 2020]
- Stochastic control and PDEs in finance: [Germain et al., 2021b]
- Stochastic control and games: [Hu and Laurière, 2023]

Thank you for your attention

Questions?

Feel free to reach out: mathieu.lauriere@nyu.edu

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5. Link with Generative Adversarial Networks

Examples

Examples

thispersondoesnotexist.com

thiscatdoesnotexist.com

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`thispersondoesnotexist.com`

`thiscatdoesnotexist.com`

[Karras et al., 2020]

GANs & MFGs

Generative Adversarial Nets [Goodfellow et al., 2014]:

Setup: data space $\mathcal{X}$ (e.g. images of fixed size); *unknown* data distribution p_{data}

Goal: be able to generate samples according p_{data}

Given: samples from data, and random noise generator p_z over some space $\mathcal{Z}$

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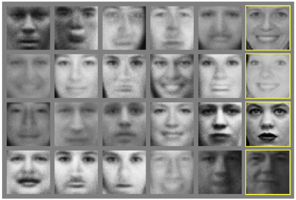

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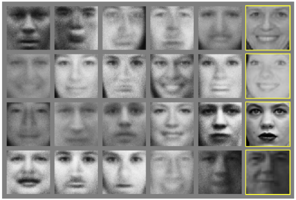

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NVIDIA'19

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Idea++: also learn $D : \mathcal{X} \to \mathbb{R}$ to distinguish between samples from $p_z \circ G^{-1}$ and p_{data}

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$$\min_\gamma \max_\delta \left\{ \mathbb{E}_{x \sim \mathbb{P}_r}[\log D_\delta(x)] + \mathbb{E}_{z \sim \mathbb{P}_z}[\log(1 - D_\delta(G_\gamma(z)))] \right\}.$$

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Variational MFG: $\inf_{u:[0,T]\times\mathbb{R}^d \to \mathbb{R}} \sup_{m:[0,T]\times\mathbb{R}^d \to \mathbb{R}} \Phi(m,u)$, where

$$\Phi(m,u) = \int_0^T \int_{\mathbb{T}^d} [m(-\partial_t u - \epsilon \Delta_x u) + m H(x, \nabla_x u, m)]\, dx dt + \int_{\mathbb{T}^d} [m(T) u(T) - m_0 u(0)]\, dx$$

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$\to$ Conceptual connection GANs/MFGs: [Cao et al., 2020]

GANs & MFGs

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$\to$ Conceptual connection GANs/MFGs: [Cao et al., 2020]

Related work: [Domingo-Enrich et al., 2020], [Lin et al., 2020], ...